Reachability Analysis in the KeYmaera X Theorem Prover

SNR 2017 | Uppsala, Sweden | April 22, 2017

Nathan Fulton

Other System Contributors: Stefan Mitsch, André Platzer, Brandon Bohrer, Yong Kiam Tan, Jan-David Quesel, ...

Trustworthy Foundations

Interactive Reachability Analysis

- Demonstration
- Bellerophon language and library

Automation and Tooling

Conclusions & Resources

KeYmaera X enables trustworthy automation for hybrid systems analysis:

- A well defined **logical foundations,**
- implemented in a **small trustworthy core**
- that ensures correctness of **automation** and **tooling**.

Trustworthy Foundations

Hybrid Programs

$a := t$

$\mathbf{a=a_0}$
$b=b_0$
$c=c_0$
...

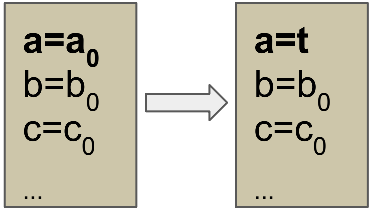
$\Rightarrow$

$\mathbf{a=t}$
$b=b_0$
$c=c_0$
...

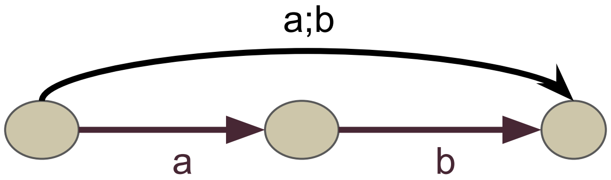
Trustworthy Foundations

Hybrid Programs

$a := t$



$a;b$



Hybrid Programs

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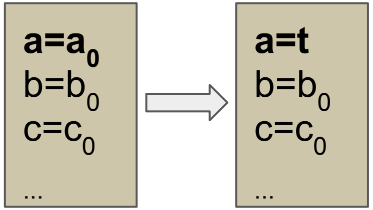

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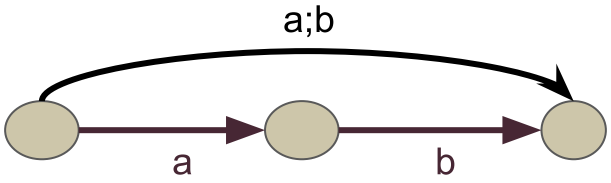

?P

If P is true: no change

If P is false: terminate

Hybrid Programs

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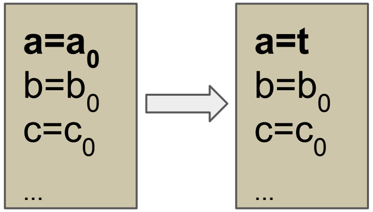

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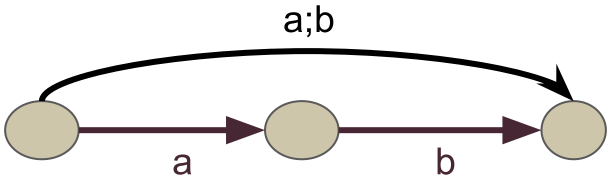
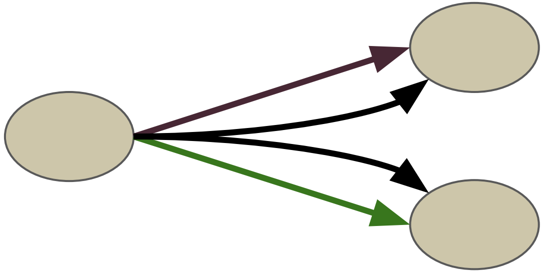


a U b



Trustworthy Foundations
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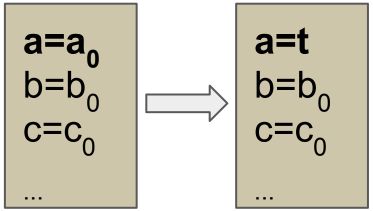
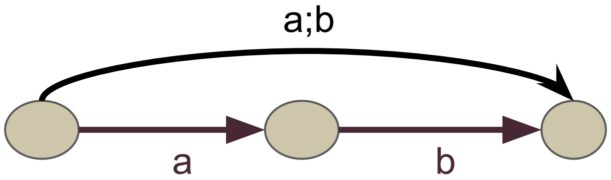


a;b



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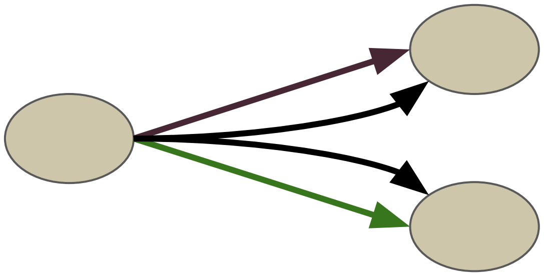
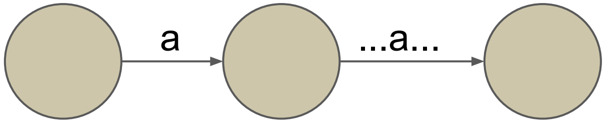


a*



Trustworthy Foundations

Hybrid Programs

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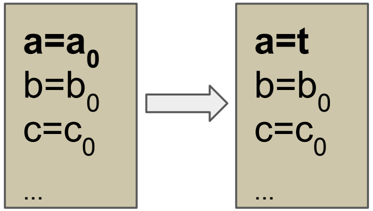

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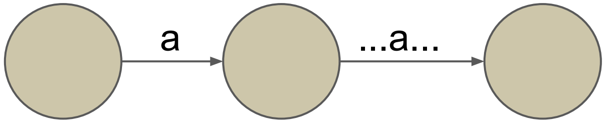

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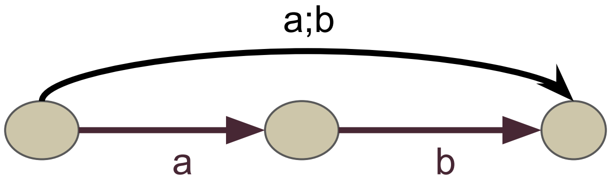
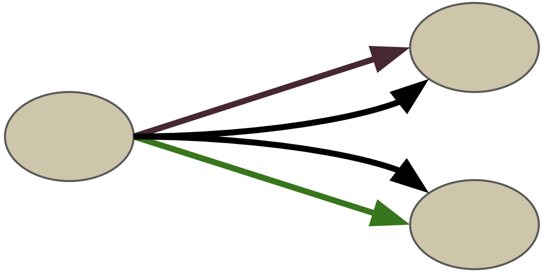
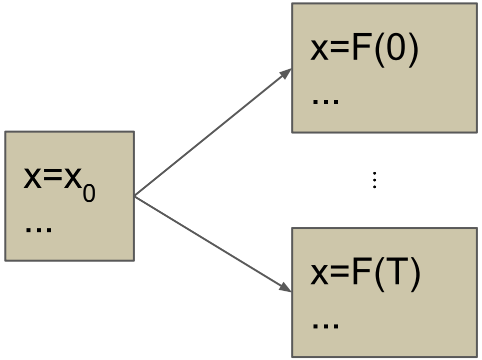


$a \cup b$

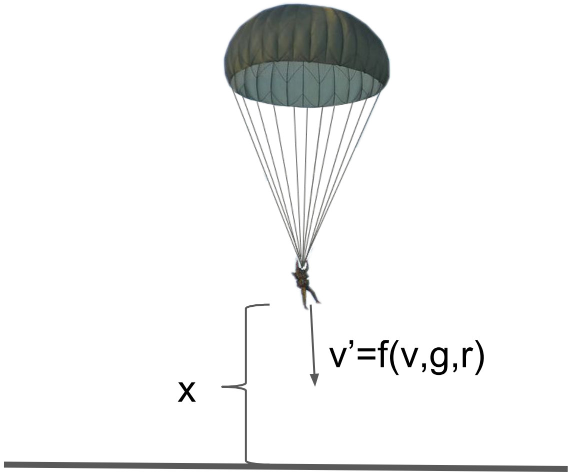


$x' = f$



Hello, World

```
{
  {?Dive ∪ r := r_p};
  t:=0;
  {x' = v,
   V' = f(v,g,r), t'=1
   & 0≤x & t≤T}
}*
```



Control: Continue diving if safe, else open parachute.
Plant: Downward velocity determined by gravity, air resistance.

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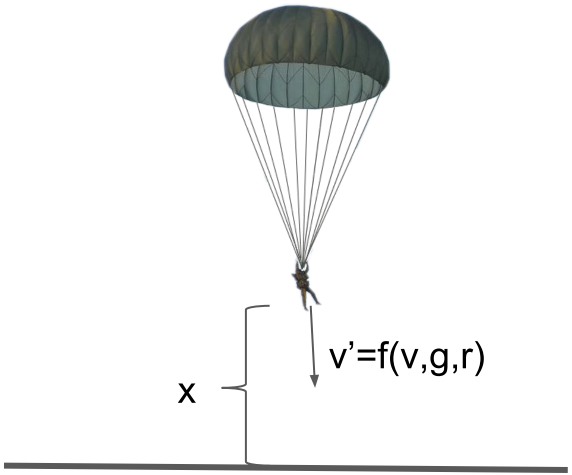

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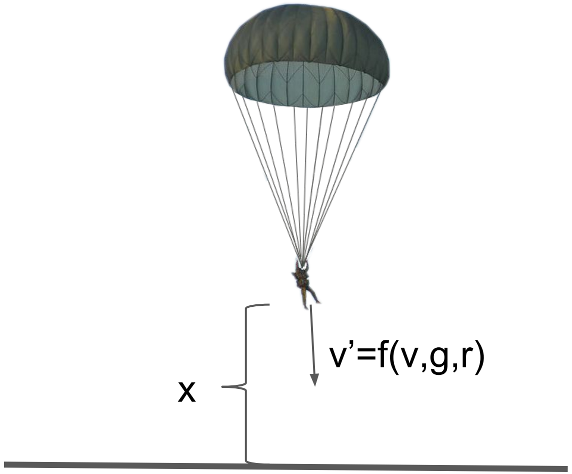
}*
```



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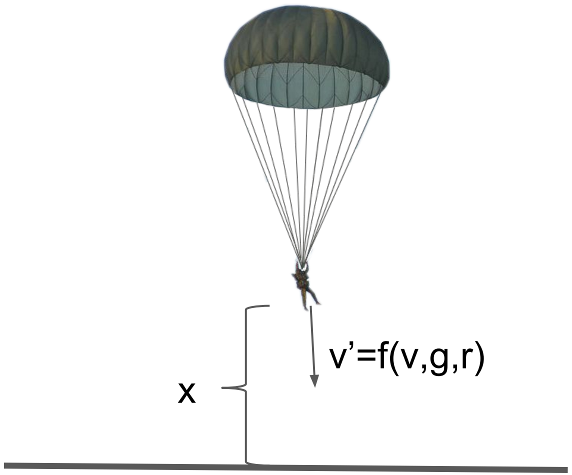
```
{
   {?Dive ∪ r := r_p};
   t:=0;
   {x′ = v,
    V′ = f(v,g,r), t′=1
    & 0≤x & t≤T}
}*
```



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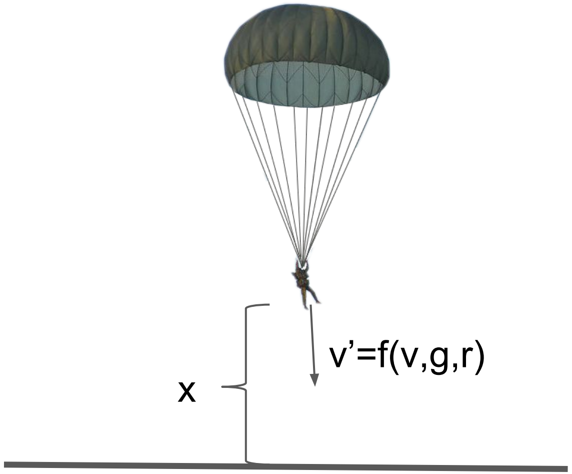

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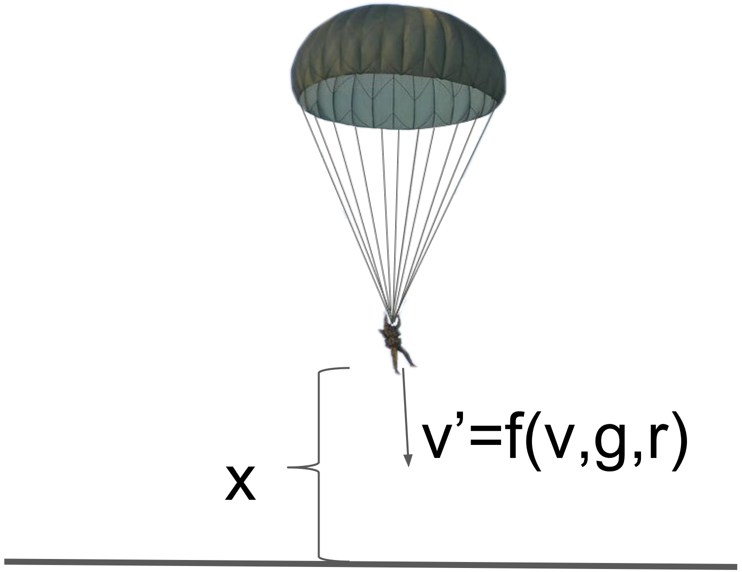
Reachability Specifications

[a]P “after every execution of a, P”

<a>P “after some execution of a, P”

Reachability Specifications

$$
\begin{aligned}
&(\mathtt{Dive}\ \&\ g>0\ \&\ \ldots)\rightarrow\\
&[\{\\
&\quad \{?\mathtt{Dive} \cup r := r_p\};\\
&\quad \{x' = v,\\
&\quad\ V' = f(v,g,r)\\
&\quad\ \&\ 0 \le x\}\\
&\}^*]\ (x=0 \rightarrow m \le v)
\end{aligned}
$$



Reachability Specifications

```
(Dive & g>0 & …)→
[{
  {?Dive ∪ r := r_p};
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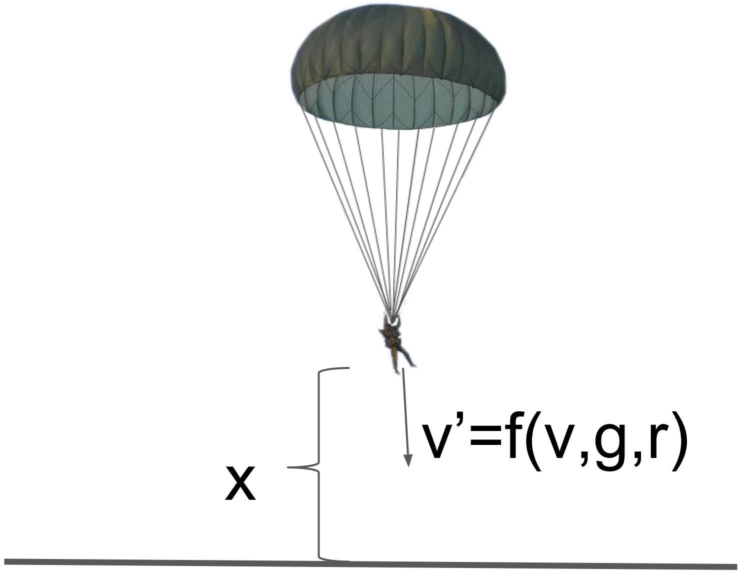
   & 0≤x}
}*] (x=0→m≤v)
```



$\underbrace{(x=0 \rightarrow m \leq v)}$

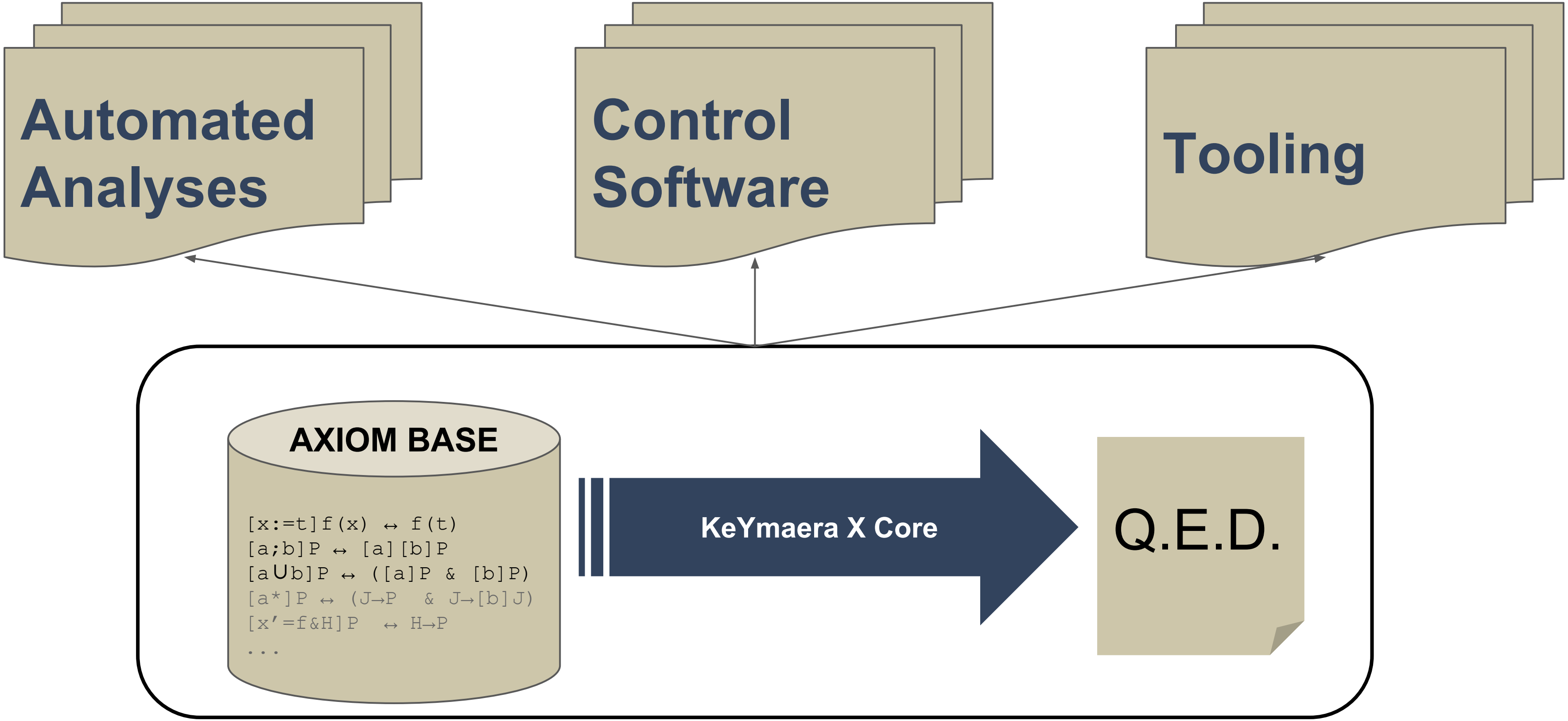
If the parachuter is on the ground, their speed is safe (m≤v≤0)

Dynamical Axioms

```
[x:=t]f(x)  ↔  f(t)
[a;b]P      ↔  [a][b]P
[a∪b]P      ↔  ([a]P & [b]P)
[a*]P       ↔  (J→P  & J→[b]J)
[x'=f&H]P   ↔  H→P
...
```

Trusted Core

Trustworthy Implementations



Prover Core Comparison

Tool	Trusted LOC (approx.)
KeYmaera X	**1,682 (out of 100,000+)**
KeYmaera	65,989
Isabelle/Pure	8,113
Coq	20,000
HSolver	20,000
dReal	50,000
SpaceEx	100,000

Interactive Reachability Analysis in KeYmaera X

KeYmaera X enables interactive verification and tool development:

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- A **standard library** of common proof techniques.

Interactive Reachability Analysis in KeYmaera X

KeYmaera X enables interactive verification and tool development:

- A **standard library** of common proof techniques.
- A **combinator language/library** for **decomposing** theorems and **composing** proof strategies.

Bellerophon

Tactic	Meaning
`prop`	Applies propositional reasoning exhaustively.
`unfold`	Symbolically executes discrete, loop-free programs.
`loop(J, i)`	Applies loop invariance axiom to position i.
`dI,dG,dC,dW`	Reasoning principles for differential equations.

Bellerophon

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100+

Interactive Reachability Analysis in KeYmaera X

Bellerophon

100+

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Combinator	Meaning
A ; B	Execute A on current goal, then execute B on the result.
$A \mid B$	Try executing A on current goal. If A fails, execute B on current goal.
A*	Run A until it no longer applies.
$A<(B_0,B_1, \ldots ,B_N)$	Execute A on current goal to create N subgoals. Run B_i on subgoal i.

Interactive Reachability Analysis in KeYmaera X

Isolating Interesting Questions

(Dive & g>0
& …)→
[{

}*] **(x=0→m≤v)**

Interactive Reachability Analysis in KeYmaera X

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prop ; loop(J,1)

(Dive &
g>0 &
…)→
J

Loop invariant holds initially

J→[

]J

Loop invariant is preserved

J →
x=0→m≤v

Loop invariant implies safety

Isolating Interesting Questions

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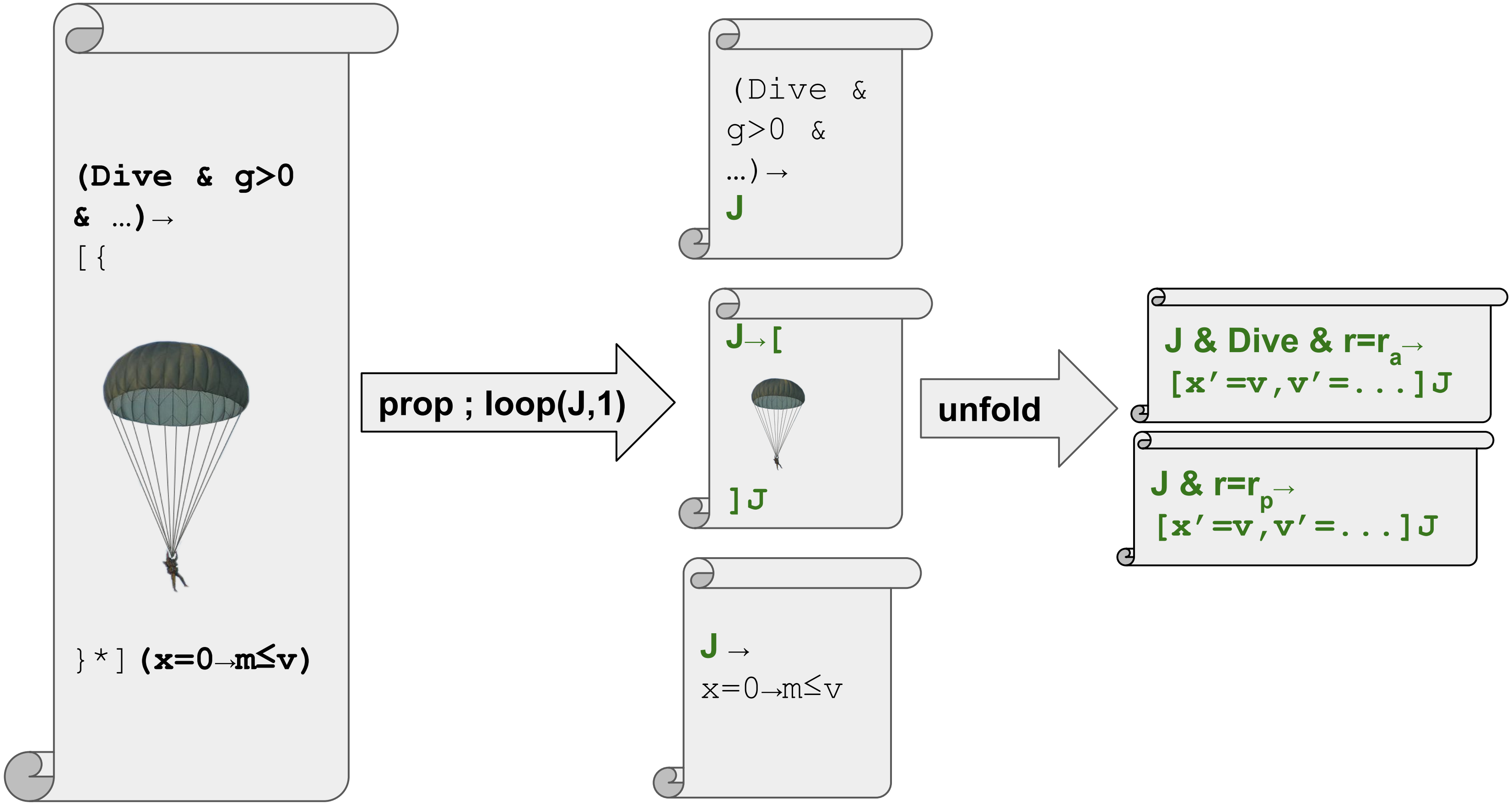
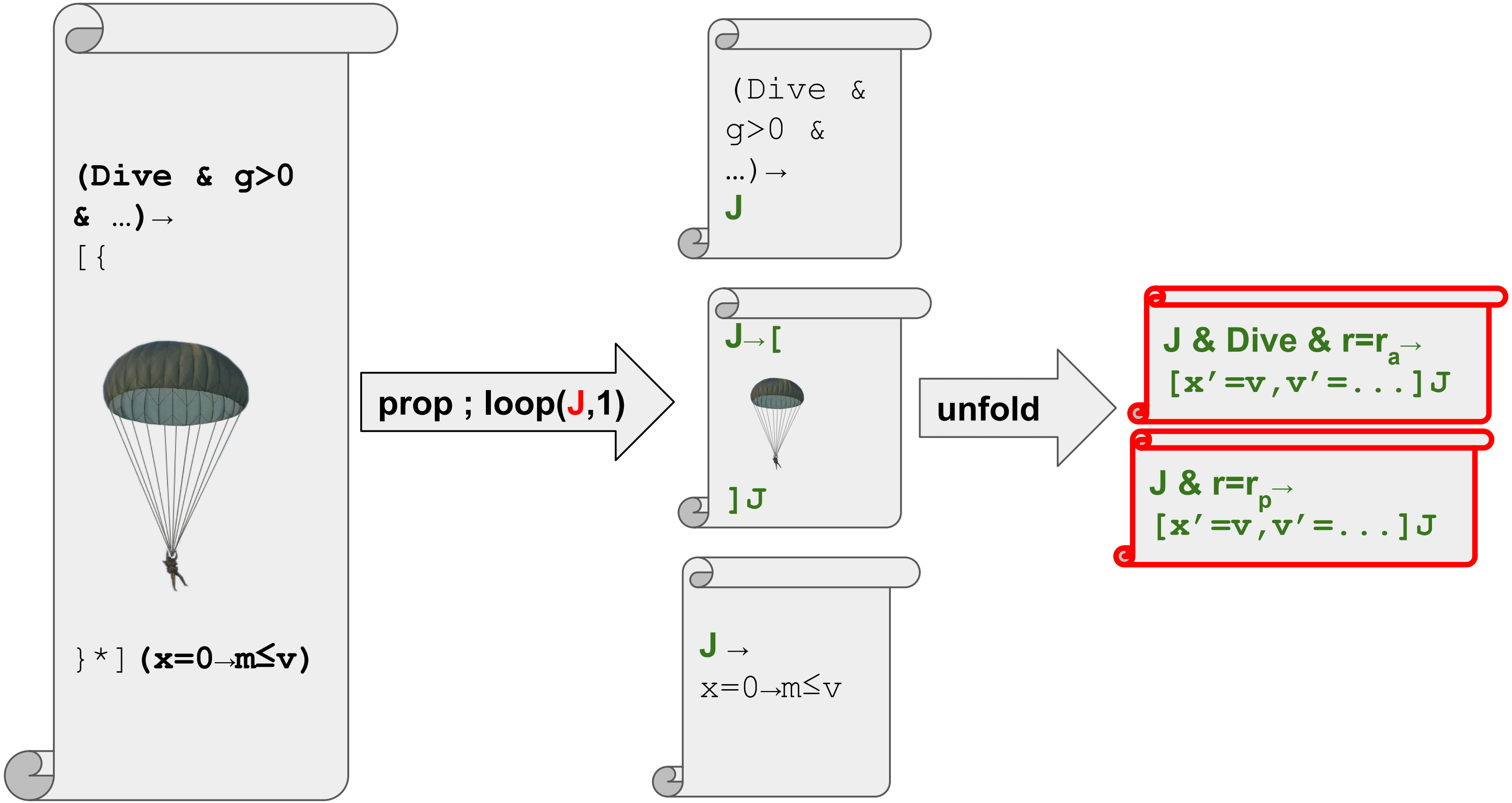


Interactive Reachability Analysis in KeYmaera X

Isolating Interesting Questions



Isolating Interesting Questions

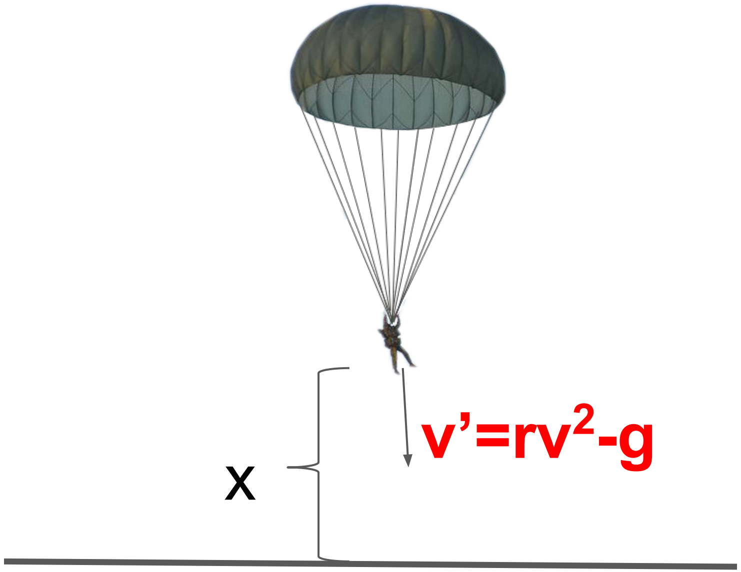
```
prop ;  loop(J, 1) <(
  QE, /* Real arith. solver */
  QE,
  Unfold <(
    … /* parachute open case */
    … /* parachute closed case */
  )
)
```

Differential Induction

`J = v > -sqrt(g/pr) > m & ...`

Parachute Open Case:

$$\underbrace{\begin{aligned} v &\geq v_0 - gt \\ &\geq v_0 - gT \\ &> -\mathrm{sqrt}(g/pr) \end{aligned}}_{\text{Inductive invariants}}$$



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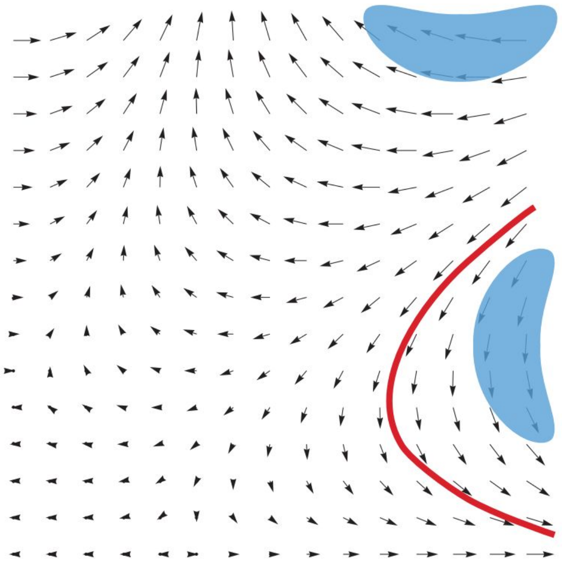
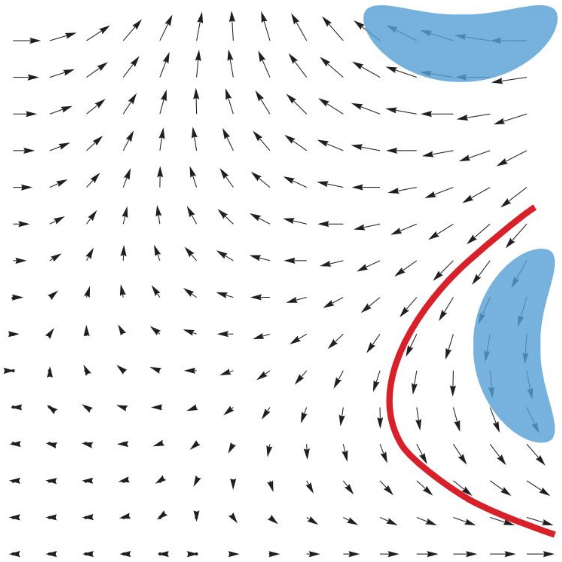


DI Axiom:

[x'=f&H]P↔(P & (H→[x':=f]P'))

Differential Induction



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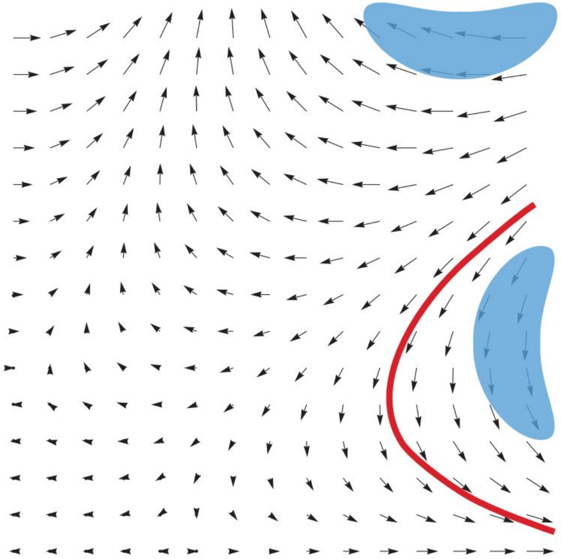
Example:

$[v' = r_p v^2 - g, t' = 1]\, v \geq v_0 - gt$

Differential Induction



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$[v'=r_p v^2-g, t'=1]v \geq v_0 - gt$ ↔

… ↔

$[v':=r_p v^2-g][t':=1]v' \geq -g*t'$ ↔

$r_p v^2-g \geq -g$ ↔

$r_p \geq 0$

Differential Induction

dI Tactic:

Side derivation:

$(v \geq v_0 - gt)' \leftrightarrow$

$(v)' \geq (v_0 - gt)' \leftrightarrow$

$(v)' \geq (v_0 - gt)' \leftrightarrow$

$(v)' \geq (v_0)' - (gt)' \leftrightarrow$

$(v)' \geq (v_0)' - (t(g)' + g(t')) \leftrightarrow$

$V' \geq v_0' - (tg' + gt')$

$H = r_p \geq 0 \ \& \ r_a \geq 0 \ \& \ g > 0 \ \& \ \ldots$

DI Axiom:

$[x' = f \& H]P \leftrightarrow (P \ \& \ (H \rightarrow [x' := f]P'))$

Example:

$[v' = r_p v^2 - g, t' = 1] v \geq v_0 - gt \leftrightarrow$

$\ldots \leftrightarrow$

$[v' := r_p v^2 - g][t' := 1] v' \geq -g * t' \leftrightarrow$

$r_p v^2 - g \geq -g \leftrightarrow$

$H \rightarrow r_p \geq 0$

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$H = r_p \geq 0$ & $r_a \geq 0$ & $g > 0$ & ...

DI Axiom:

$[x'=f\&H]P \leftrightarrow (P \;\&\; (H \rightarrow [x':=f]P'))$

Example:

$[v'=r_p v^2 - g, t'=1]v \geq v_0 - gt \leftrightarrow$

... $\leftrightarrow$

$[v':=r_p v^2 - g][t':=1]v' \geq -g*t' \leftrightarrow$

$r_p v^2 - g \geq -g \leftrightarrow$

$H \rightarrow r_p \geq 0$

Tactics recover a useful level of abstraction.

Pedantry is the price of trust.

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Bellerophon automates pedantic deductions.

Automation and Tooling

Hybrid Systems Analyses can be built on top of KeYmaera X.

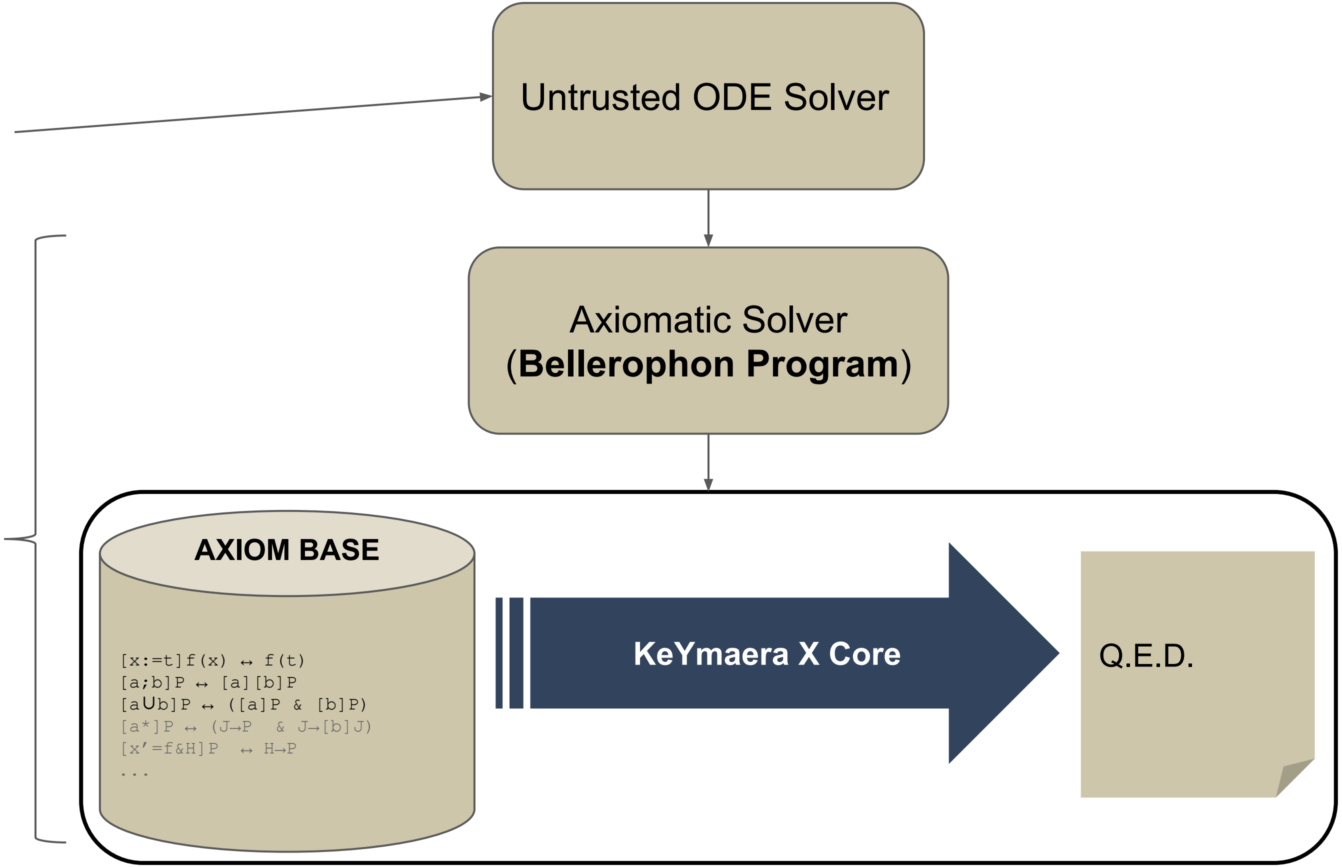
Examples:

- ODE Solver
- Runtime Monitoring

Toward Automated Deduction

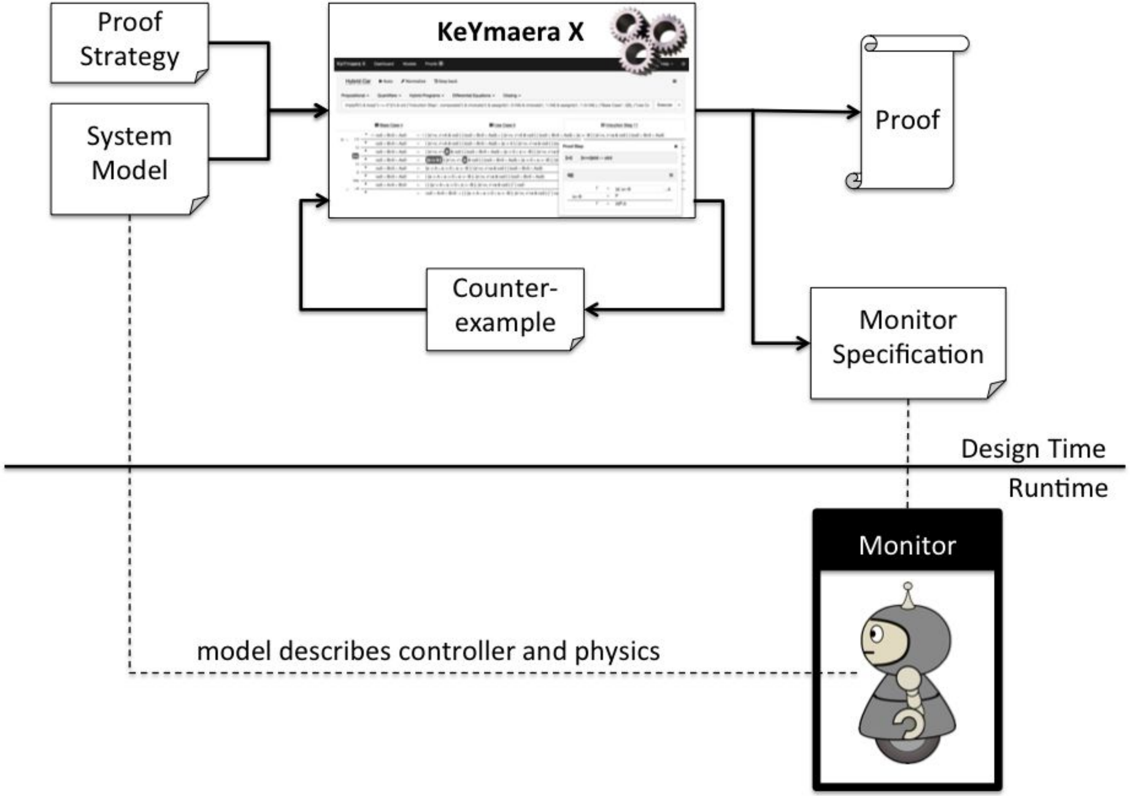
Solving Differential Equations

1. Use untrusted code to find a conjecture.
2. Prove the conjecture systematically.

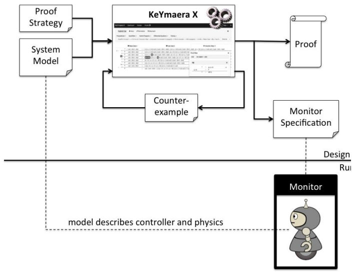


Toward Automated Deduction

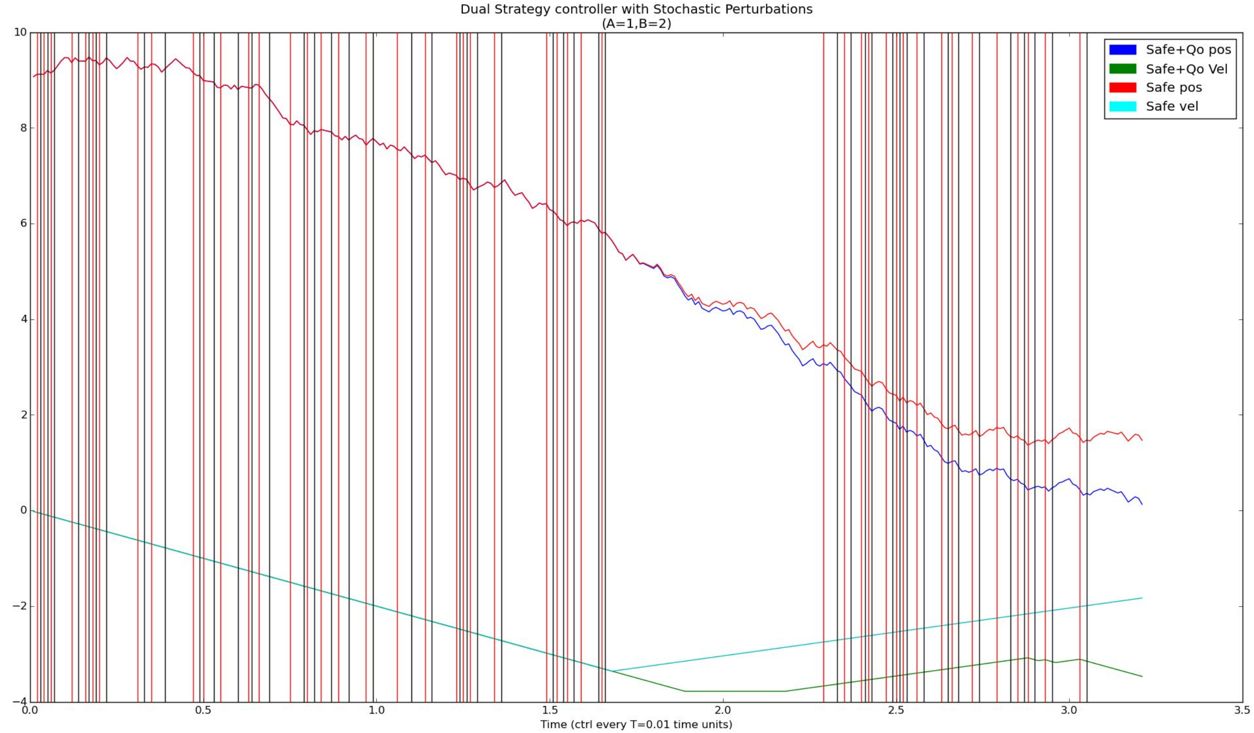
ModelPlex Tactic



Learning how to be Safe



=>



Other Proof Automation & Tooling

- Automated Analysis for nonlinear systems:
  - Pretty decent automation for systems with univariate nonlinearities.
  - Heuristics for multi-variate systems.

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- ...
- **Component-based Verification Tooling**

  Mueller et al., *Change and Delay Contracts for Hybrid System Component Verification*, **FASE'17 -- Thursday 10:30-12:30**

Conclusion

KeYmaera X is a hybrid systems theorem prover with:

- A small and trustworthy prover core and
- Excellent infrastructure for **interactively verifying complex systems** and **implementing automated analyses**.

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Project Website (start here)	keymaeraX.org
Online Demo	web.keymaeraX.org
GPL'd Source Code	github.com/ls-lab/KeYmaeraX-release
Course Materials	symbolaris.com/course/fcps17.html

Developers:

- Stefan Mitsch
- Nathan Fulton
- Andre Platzer
- Jan-David Quesel
- Brandon Bohrer
- Yong Kiam Tan
- Markus Voelp

Special Thanks:

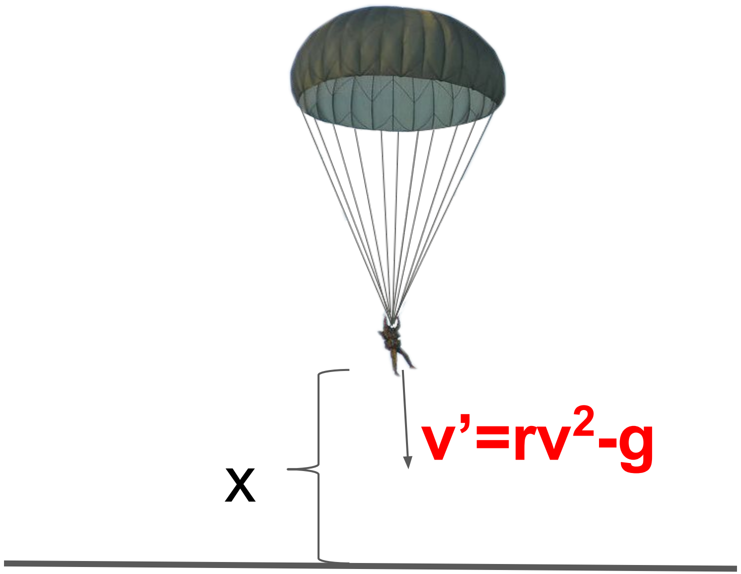
- 15-424 students, Jean-Baptiste Jeanin, Khalil Ghorbal, Daniel Ricketts

Differential Ghosts

Parachute Closed:

$$J \;\&\; t=0 \;\&\; r=r_p \rightarrow [x'=v, v'=rv^2-g \;\&\; 0\leq x \;\&\; t\leq T]\, v>-\mathrm{sqrt}(g/pr) > m$$

Proof requires a **differential ghost** because the property is **not inductive**.



Differential Ghosts

An example differential ghost.

```
x>0 → [x'=-x]x>0
```

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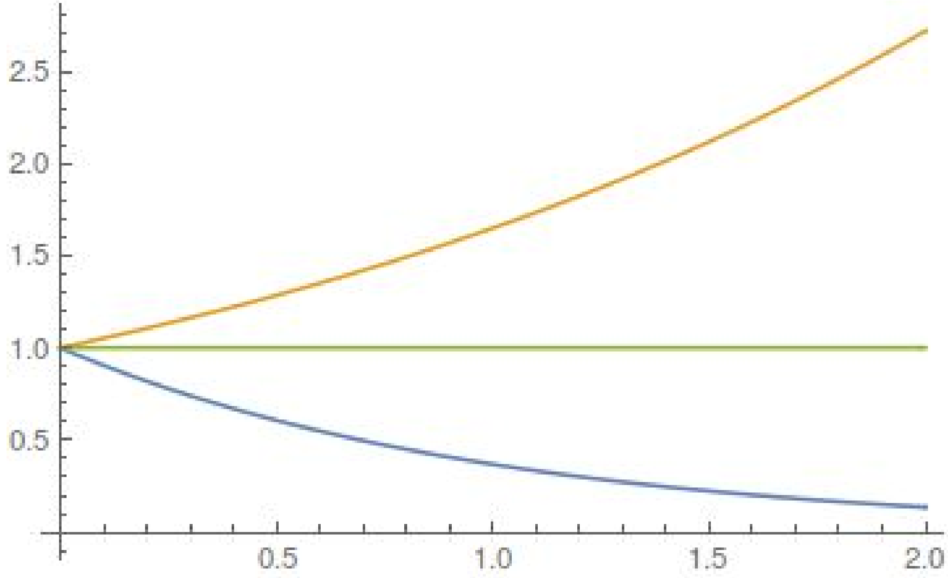
```
x>0 → [x'=-x]x>0
Ghost:      y'=y/2
Conserved: 1=xy²
```



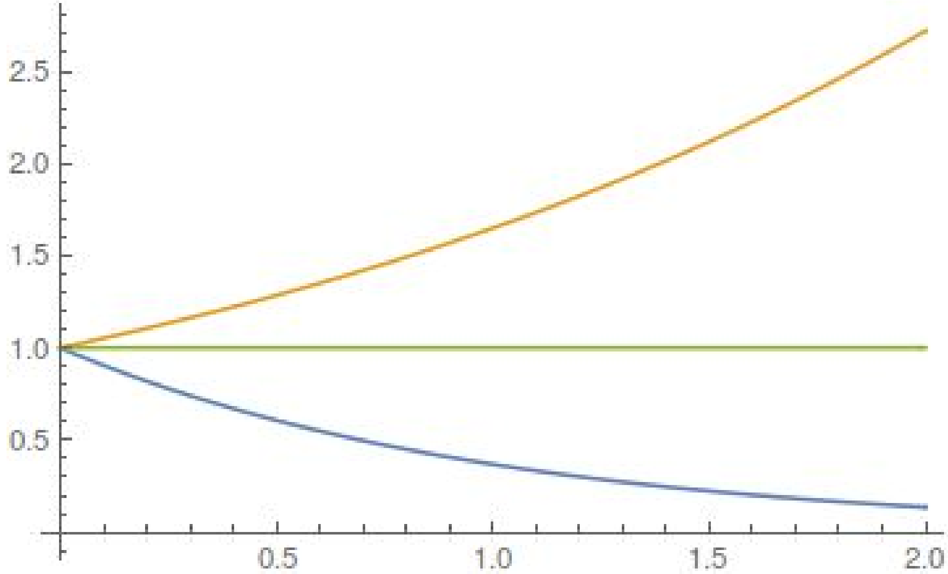
Differential Ghosts

An example differential ghost.

$x>0 \rightarrow [x'=-x]x>0$

Ghost: $y'=y/2$

Conserved: $1=xy^2$



Notice:

$x>0 \leftrightarrow \exists y.1=xy^2$

Therefore, suffices to show:

$1=xy^2 \rightarrow \exists y.[x'=-x, y'=y/2]1=xy^2$