A Logic of Proofs for Differential Dynamic Logic

Toward Independently Checkable Proof Certificates for Differential Dynamic Logic

Nathan Fulton   Andrè Platzer
Carnegie Mellon University
CPP'16

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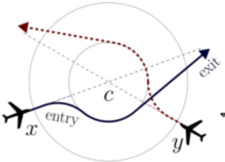
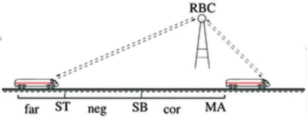
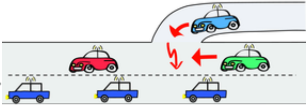
Motivation

Strong evidence that Cyber-Physical Systems are safe.







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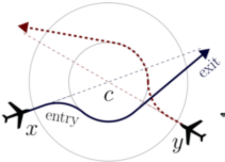
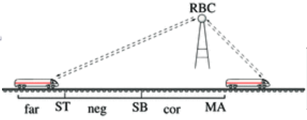
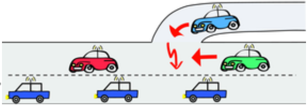
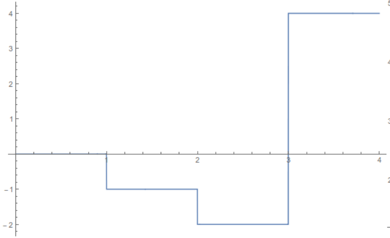
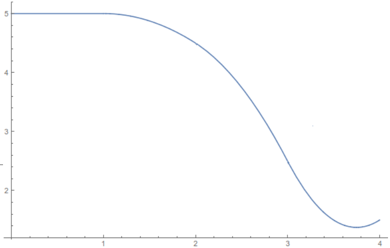


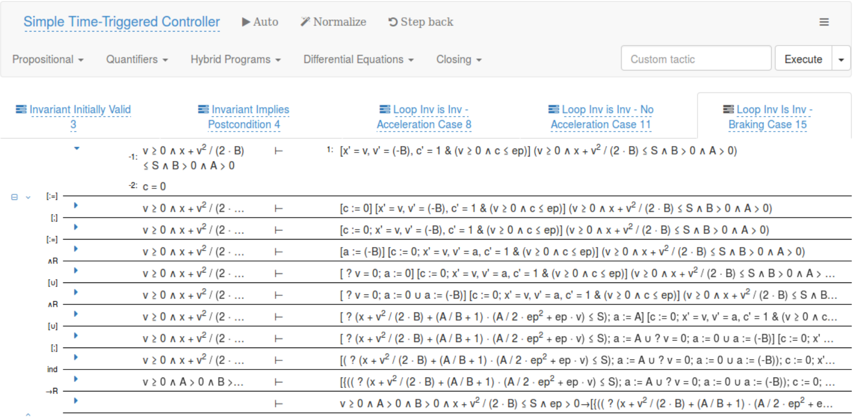








KeYmaera X



Simple Time-Triggered Controller
▶ Auto
Normalize
Step back
Propositional
Quantifiers
Hybrid Programs
Differential Equations
Closing
Custom tactic
Execute
Invariant Initially Valid 3
Invariant Implies Postcondition 4
Loop Inv is Inv - Acceleration Case 8
Loop Inv is Inv - No Acceleration Case 11
Loop Inv Is Inv - Braking Case 15
-1: v ≥ 0 ∧ x + v² / (2 · B) ≤ S ∧ B > 0 ∧ A > 0
-2: c = 0
⊢
1: [x' = v, v' = (-B), c' = 1 & (v ≥ 0 ∧ c ≤ ep)] (v ≥ 0 ∧ x + v² / (2 · B) ≤ S ∧ B > 0 ∧ A > 0)
[:=]
v ≥ 0 ∧ x + v² / (2 · … ⊢ [c := 0] [x' = v, v' = (-B), c' = 1 & (v ≥ 0 ∧ c ≤ ep)] (v ≥ 0 ∧ x + v² / (2 · B) ≤ S ∧ B > 0 ∧ A > 0)
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∧R
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[∪]
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∧R
v ≥ 0 ∧ x + v² / (2 · … ⊢ [? (x + v² / (2 · B) + (A / B + 1) · (A / 2 · ep² + ep · v) ≤ S); a := A] [c := 0; x' = v, v' = a, c' = 1 & (v ≥ 0 ∧ c…
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ind
v ≥ 0 ∧ A > 0 ∧ B >… ⊢ [{((? (x + v² / (2 · B) + (A / B + 1) · (A / 2 · ep² + ep · v) ≤ S); a := A ∪ ? v = 0; a := 0 ∪ a := (-B)); c := 0; …
→R
⊢ v ≥ 0 ∧ A > 0 ∧ B > 0 ∧ x + v² / (2 · B) ≤ S ∧ ep > 0→[{((? (x + v² / (2 · B) + (A / B + 1) · (A / 2 · ep² + e…

Criteria for Evidence of a Successful Verification Effort

- ☑ Hybrid Systems Proofs (via KeYmaera X)
- ☐ Persistent – truth-preservation is insufficient!
- ☐ Permanent – Tactics are not proofs
- ☐ Portable – Between machines, between logics

Approach

$$e : \phi$$

Approach

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Outline:

- The Language of Differential Dynamic Logic
- Uniform Substitution Calculus of d$\mathcal{L}$
- LPd$\mathcal{L}$

Hybrid Programs Model Cyber-Physical Systems

Definition (Hybrid Programs)

Assign	$x := \theta$
Test	$?\varphi$
Sequence	$\alpha; \beta$
Choice	$\alpha \cup \beta$
Iteration	α^*

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Definition (Hybrid Programs)

Assign	$x := \theta$
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Sequence	$\alpha; \beta$
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Iteration	α^*
ODEs	$\{x_1' = \theta_1, \dots, x_n' = \theta_n \& \varphi\}$

Example

$$\Big(\underbrace{(acc := A \cup acc := 0)}_{\textit{Control}}\ ;\ \underbrace{\{pos' = vel,\ vel' = acc\}}_{\textit{Physical System Model}}\Big)^*$$

d$\mathcal{L}$

FOL over Real Closed Fields $+ [\alpha]\varphi + \langle\alpha\rangle\varphi$

Example

$$\underbrace{vel \geq 0 \land A > 0}_{\text{initial condition}} \rightarrow$$

$$[(\underbrace{(acc := A \cup acc := 0)}_{ctrl}\ ;\ \underbrace{\{pos' = vel,\ vel' = acc\}}_{plant})^*]\ \underbrace{vel \geq 0}_{\text{postcondition}}$$

Deduction in Differential Dynamic Logic

$$\dfrac{v \geq 0, z < m \vdash \forall t \geq 0 [z := -\dfrac{b}{2}t^2 + vt + z] z \leq m}{v \geq 0, z < m \vdash [z' = v, v' = -b] z \leq m} \textsc{DiffSolve}$$

Uniform Substitution Isolates Binding Structure

DiffSolve as a single axiom:

$[x' = f \& q(x)]p(x) \leftrightarrow \forall t \geq 0((\forall 0 \leq s \leq t q(x+fs)) \rightarrow [x := x+ft]p(x))$

Sound **uniform substitutions** are used in deductions:

$$\frac{\varphi}{\sigma(\varphi)}\mathrm{US}$$

Significant Features of d$\mathcal{L}$

$$\textsc{BoxChoice}\quad \frac{\Gamma \vdash [\alpha]\varphi \qquad \Gamma \vdash [\beta]\varphi}{\Gamma \vdash [\alpha \cup \beta]\varphi}$$

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$\sigma =$

$$a \leadsto x := 4$$

$$b \leadsto x := 5$$

$$p(?) \leadsto x > 3$$

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Contribution: A Logic of Proofs for d$\mathcal{L}$

LPd$\mathcal{L}$ extends the grammar of d$\mathcal{L}$ with formulas of the form

$$\underbrace{e}_{\textbf{LPd}\mathcal{L}\textbf{ proof term}} : \underbrace{\varphi}_{\mathrm{d}\mathcal{L}\textit{formula}}$$

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$\langle e, d \ \rangle ::= \ c_\phi$

Example (Proof Constants)

$$(i_{[:=]}) : ([x := t]p(x) \leftrightarrow p(t))$$
$$(j_{x>y \wedge y>z \rightarrow x>z}) : (x > y \wedge y > z \rightarrow x > z)$$

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$$\begin{aligned} \langle e, d \rangle ::= \;& c_\phi \\ \mid \;& e \wedge d \end{aligned}$$

Example (Conjunctions)

$$(i_{:=} \wedge j_{x>0}) : (([x := t]p(x) \leftrightarrow p(t)) \wedge x > 0)$$

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Example ($\bullet$)

If

$$e : \varphi \rightarrow \psi \tag{1}$$
$$d : \varphi \tag{2}$$

Then $e \bullet d : \psi$.

Directional application performs a similar operation on equivalences.

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Example (Uniform Substitution of Axiom $[x := t]p(x) \leftrightarrow p(t)$)

$$\sigma_{\{t \mapsto 0\ ,\ p(\cdot) \mapsto \cdot \geq 0\}}(i_{[:=]}) : [x := 0]x \geq 0 \leftrightarrow 0 \geq 0$$

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$$\begin{aligned}
\langle e, d\,\rangle \;::=\;& c_\phi \\
\mid\;& e \wedge d \\
\mid\;& e \bullet d \mid e \bullet_\leftarrow d \mid e \bullet_\rightarrow d \\
\mid\;& \sigma e \mid \mathscr{B} e \\
\mid\;& \mathrm{CT}_\sigma e \mid \mathrm{CQ}_\sigma e \mid \mathrm{CE}_\sigma e
\end{aligned}$$

Example (US Instances of Proof Rules)

$CE_{\{t \rightsquigarrow 0\ ,\ p(\cdot) \rightsquigarrow \cdot \geq 0\}}\, i_{[x:=t]p(t) \leftrightarrow p(x)}$:

$$([\{z' = a\}][x := 0]x \geq 0) \leftrightarrow ([\{z' = a\}]0 \geq 0)$$

Sampling of Axioms and Proof Rules

$$\phi \qquad \text{(d}\mathcal{L}\text{ Axiom)}$$

$$i_A : A \qquad \text{(d}\mathcal{L}\text{ Constants)}$$

$$\frac{e : \phi \qquad d : \psi}{(e \wedge d) : (\phi \wedge \psi)} \qquad \text{(And)}$$

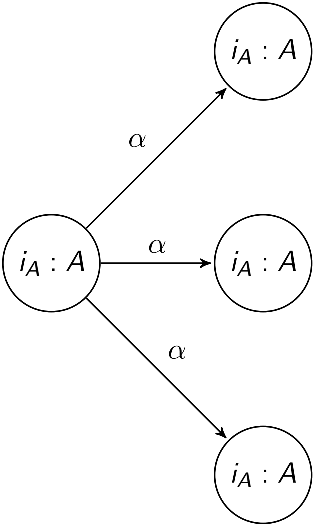
$$\frac{e : (\phi \rightarrow \psi) \qquad d : \phi}{e \bullet d : \psi} \qquad \text{(Application)}$$

$$\frac{e : \phi}{\sigma e : \sigma(\phi)} \qquad \text{(US Proof Term)}$$

$$\frac{\sigma e : \sigma(p(\bar{x}) \leftrightarrow q(\bar{x}))}{\mathrm{CE}_\sigma e : \sigma(C(p(\bar{x}) \leftrightarrow C(q(\bar{x})))} \qquad (\mathrm{CE}_\sigma)$$

Only side-condition: admissibility of σs.

Semantics of LPd$\mathcal{L}$



- $[\![\phi]\!]^I = [\![\phi]\!]^I_{\mathsf{d}\mathcal{L}}$
- $[\![i_A : A]\!]^I = S$ for d$\mathcal{L}$ axioms A
- $[\![j_T : T]\!] = S$ for $\mathsf{FOL}_{\mathbb{R}}$ tautologies T
- $[\![e \wedge d : \phi \wedge \psi]\!]^I =$ $[\![e : \phi]\!]^I \cap [\![d : \psi]\!]^I$
- $[\![e \bullet d : \phi]\!]^I =$ $\bigcup_\psi [\![e : (\psi \to \phi)]\!]^I \cap [\![d : \psi]\!]^I$
- ...

Correctness Properties

Theorem (Proof terms justify theorems)

Let e be a proof term and ϕ a d$\mathcal{L}$ *formula. If* $\vdash_{\mathsf{LPd}\mathcal{L}} e : \phi$ *then* $\vdash \phi$.

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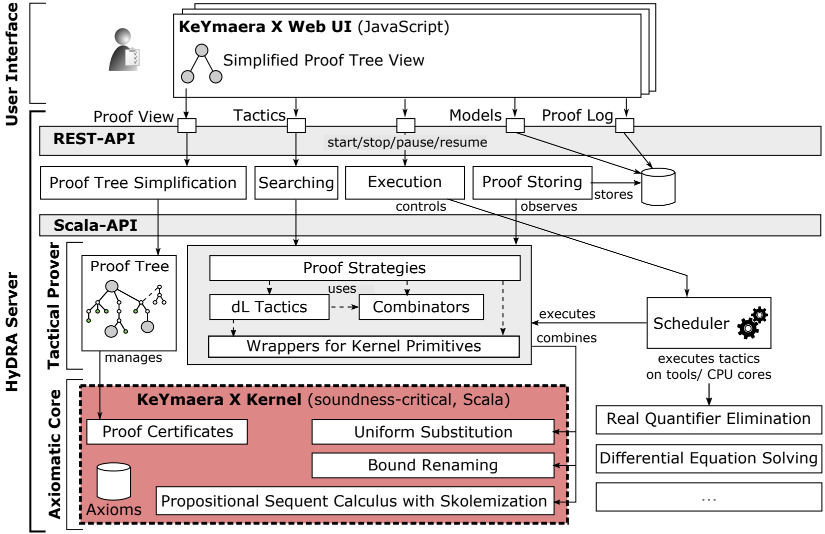

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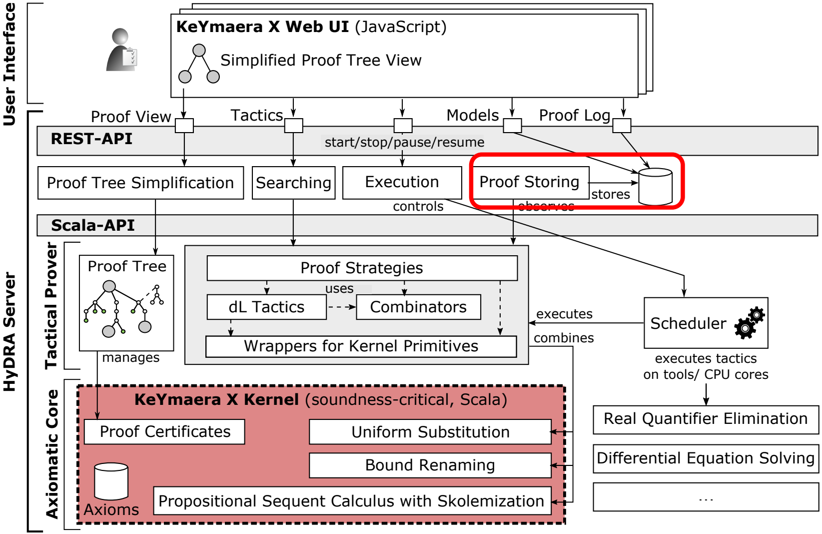

Adding Proof Terms Without Adding Soundness-Critical Code

Proof.

Case σe**.** Suppose that $\vdash_{\mathsf{LPd}\mathcal{L}} \sigma e : \phi$. By [a lemma], $\phi = \sigma(\phi')$ and $\vdash_{\mathsf{LPd}\mathcal{L}} e : \phi'$ for some ϕ'. The induction hypothesis for the smaller proof term e gives $\vdash_{\mathsf{d}\mathcal{L}} \phi'$. Therefore, $\vdash_{\mathsf{d}\mathcal{L}} \sigma(\phi')$ (i.e., ϕ) is provable by US. $\square$

```
def ProofChecker(e : ProofTerm, phi: Formula) = ...
  case UsubstTerm(e, phiPrime, usubst) => {
    val phiPrimeCert = ProofChecker(e, phiPrime)
    Provable.startProof(phi)
      .(UniformSubstitutionRule(
          usubst,
          phiPrime), 0)
      .(phiPrimeCert, 0)
  }
```

Ongoing Work

- Controller Synthesis from Non-deterministic Models
- A proof term construction semantics for the Bellerophon tactics language of KeYmaera X

Conclusion

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and furthermore **reifies** the structure of proofs

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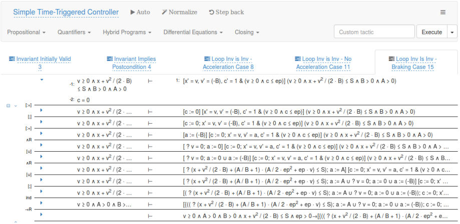
and furthermore **reifies** the structure of proofs
by **parsimoniously extending** existing theory and implementation.

$$
\begin{array}{rl}
\text{DW} & [x' = f(x) \,\&\, q(x)]q(x) \\
\text{DC} & \big([x' = f(x) \,\&\, q(x)]p(x) \leftrightarrow [x' = f(x) \,\&\, q(x) \wedge r(x)]p(x)\big) \leftarrow [x' = f(x) \,\&\, q(x)]r(x) \\
\text{DE} & [x' = f(x) \,\&\, q(x)]p(x, x') \leftrightarrow [x' = f(x) \,\&\, q(x)][x' := f(x)]p(x, x') \\
\text{DI} & [x' = f(x) \,\&\, q(x)]p(x) \leftarrow \big(q(x) \rightarrow p(x) \wedge [x' = f(x) \,\&\, q(x)](p(x))'\big) \\
\text{DG} & [x' = f(x) \,\&\, q(x)]p(x) \leftrightarrow \exists y\, [x' = f(x), y' = a(x)y + b(x) \,\&\, q(x)]p(x) \\
\text{DS} & [x' = f \,\&\, q(x)]p(x) \leftrightarrow \forall t{\geq}0\, \big((\forall 0{\leq}s{\leq}t\, q(x + fs)) \rightarrow [x := x + ft]p(x)\big) \\
[' :=] & [x' := f]p(x') \leftrightarrow p(f) \\
+' & (f(\bar{x}) + g(\bar{x}))' = (f(\bar{x}))' + (g(\bar{x}))' \\
\cdot' & (f(\bar{x}) \cdot g(\bar{x}))' = (f(\bar{x}))' \cdot g(\bar{x}) + f(\bar{x}) \cdot (g(\bar{x}))' \\
\circ' & [y := g(x)][y' := 1]\big((f(g(x)))' = (f(y))' \cdot (g(x))'\big)
\end{array}
$$



keymaeraX.org · github.com/LS-Lab/KeYmaeraX-release
nfulton@nfulton.org