Bellerophon: Tactical Theorem Proving for Hybrid Systems

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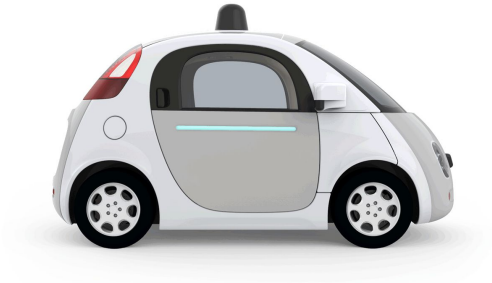
Carnegie Mellon University

Cyber-Physical Systems

Cyber-Physical Systems combine computation and control.



Hybrid Systems model combinations of discrete and continuous dynamics.

Bellerophon

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Verifying hybrid systems is hard.
Bellerophon demonstrates how to tackle hybrid systems with tactics:

- Build on a sound core.
- Implement high-level primitives for hybrid systems proofs.
- Automate common constructions (for ODEs and control software)

Bellerophon

Theorem	Bellerophon LOC	Conceptual Proof Steps	Hybrid Systems Axiom Applications
Static Safety	12	71	30,355
Passive-Friendly Safety	45	140	68,620
Orientation Safety	15	108	173,989
Pass Intersection Liveness	234	440	61,878

KeYmaera X: Trustworthy Foundations

Interactive Reachability Analysis

- Bellerophon combinator language
- Bellerophon standard library for hybrid systems
- Demonstration

Bellerophon for Automation and Tooling

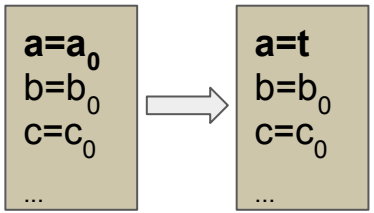
Conclusions & Resources

KeYmaera X enables trustworthy automation for hybrid systems analysis:

- A well-defined **logical foundations,**
- implemented in a **small trustworthy core**
- that ensures correctness of **automation** and **tooling**.

Hybrid Programs

a := t



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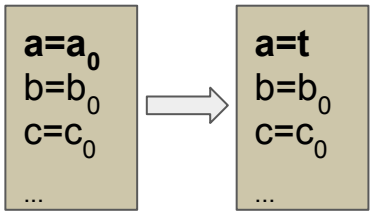
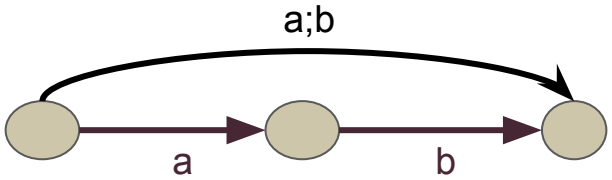


a;b



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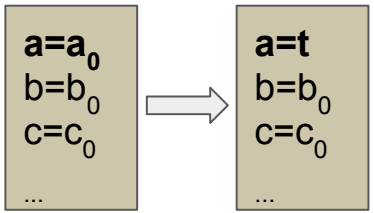

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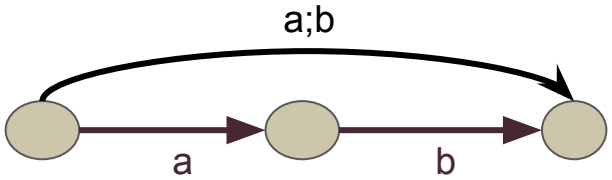

?P

If P is true: no change

If P is false: terminate

Hybrid Programs

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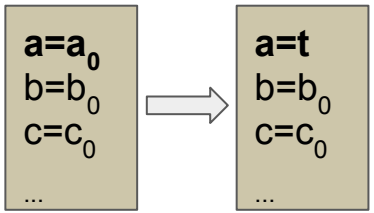

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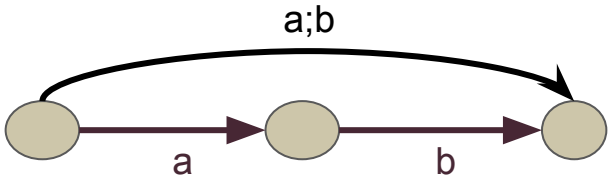
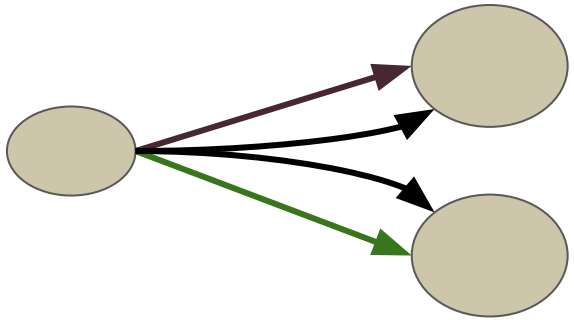


a ∪ b



Hybrid Programs

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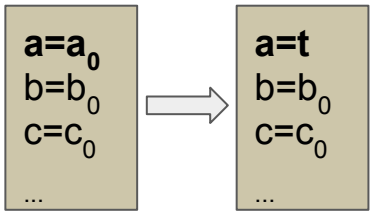

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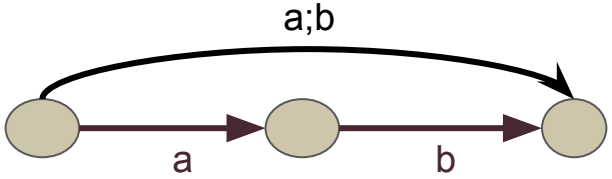
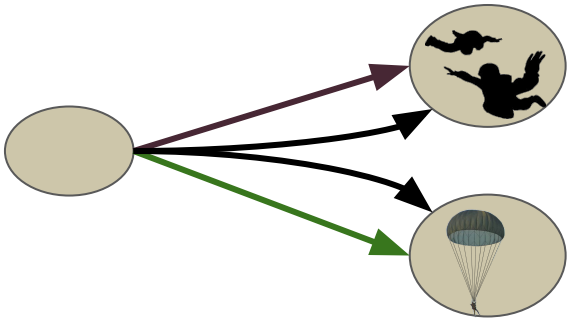


$a \cup b$



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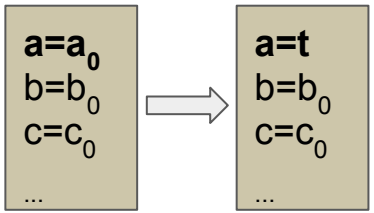

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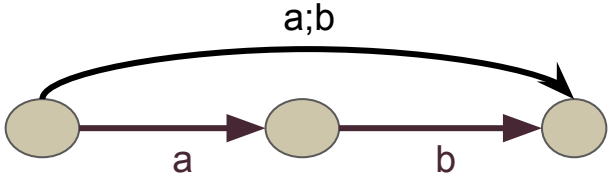

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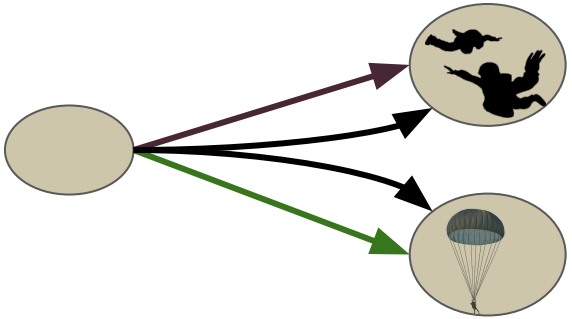
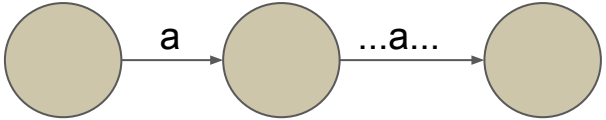


a^*



Hybrid Programs

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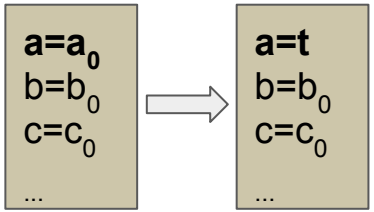

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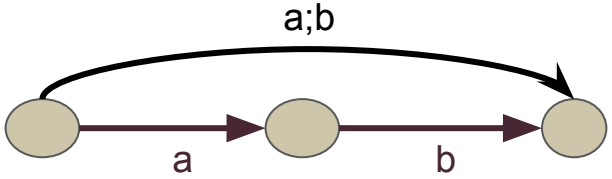

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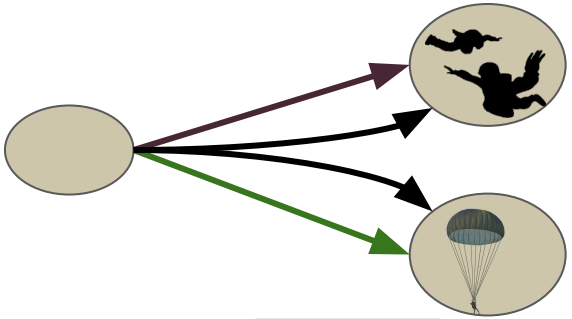
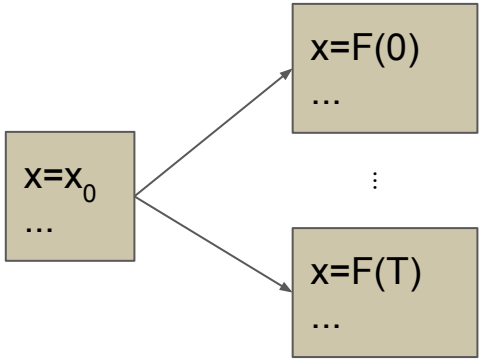


x’=f



Reachability Specifications

$[a]P$ “after every execution of a, P”

$\langle a \rangle P$ “after some execution of a, P”

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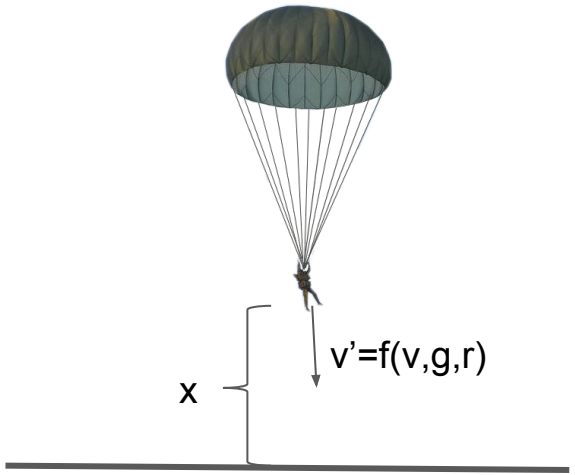
$$\text{init} \rightarrow [\{x := u(x); x' = f(x)\}^*]\text{safe}$$

Hello, World

$$
\{ \\
\quad \{?\text{Dive} \cup r := r_p\}; \\
\quad t:=0; \\
\quad \{x' = v, \\
\quad\ V' = f(v,g,r), \ t'=1 \\
\quad\ \& \ 0 \le x \ \& \ t \le T\} \\
\}^*
$$



Control: Continue diving if safe, else open parachute.
Plant: Downward velocity determined by gravity, air resistance.

Hello, World

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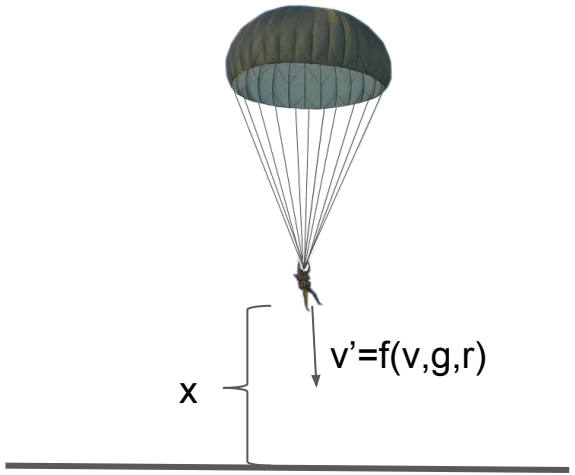

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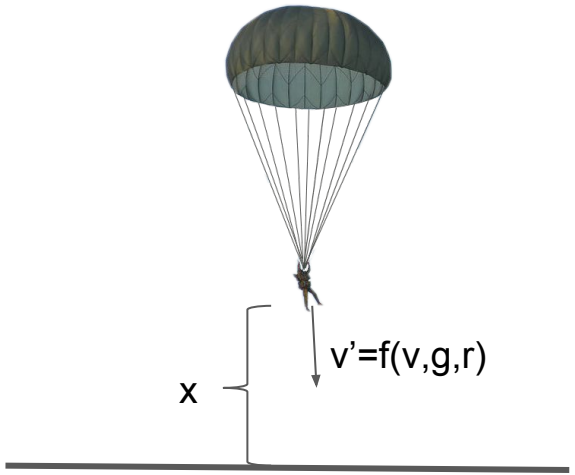

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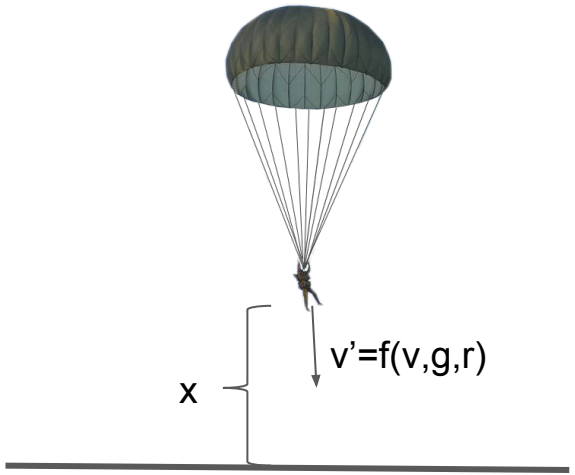

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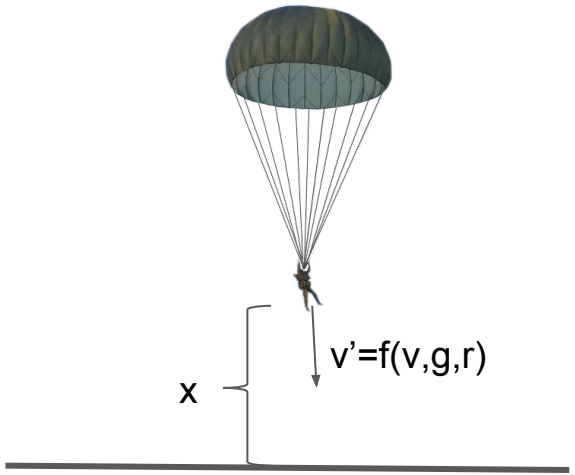

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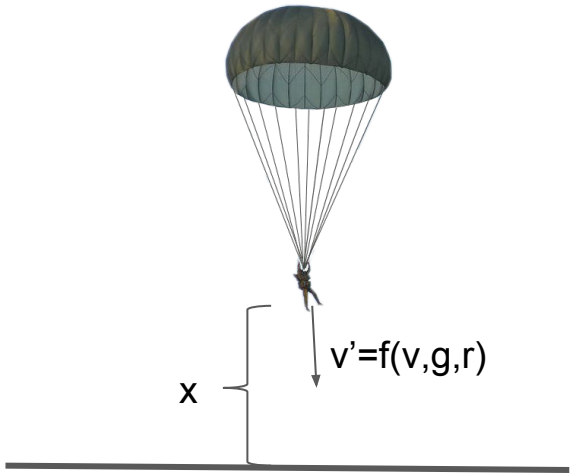

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Reachability Specifications

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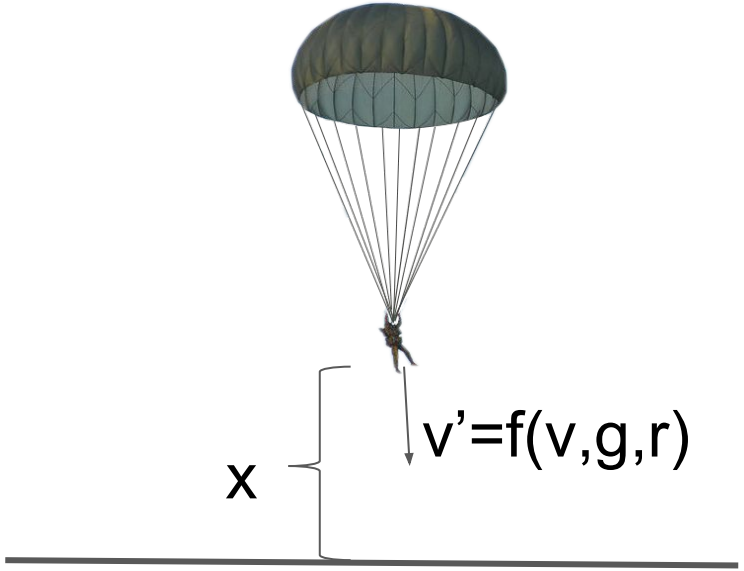
$$\quad \{x' = v,$$
$$\quad\; V' = f(v,g,r)$$
$$\quad\; \& \; 0 \leq x\}$$
$$\}^*] (x=0 \rightarrow m \leq v)$$



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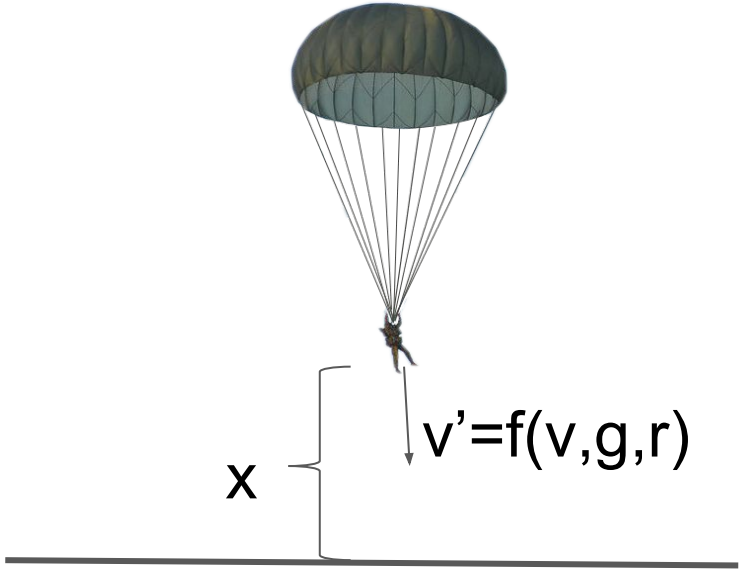


If the parachuter is on the ground, their speed is safe ($m \leq v \leq 0$)

Dynamical Axioms

$$[x:=t]f(x) \leftrightarrow f(t)$$

$$[a;b]P \leftrightarrow [a][b]P$$

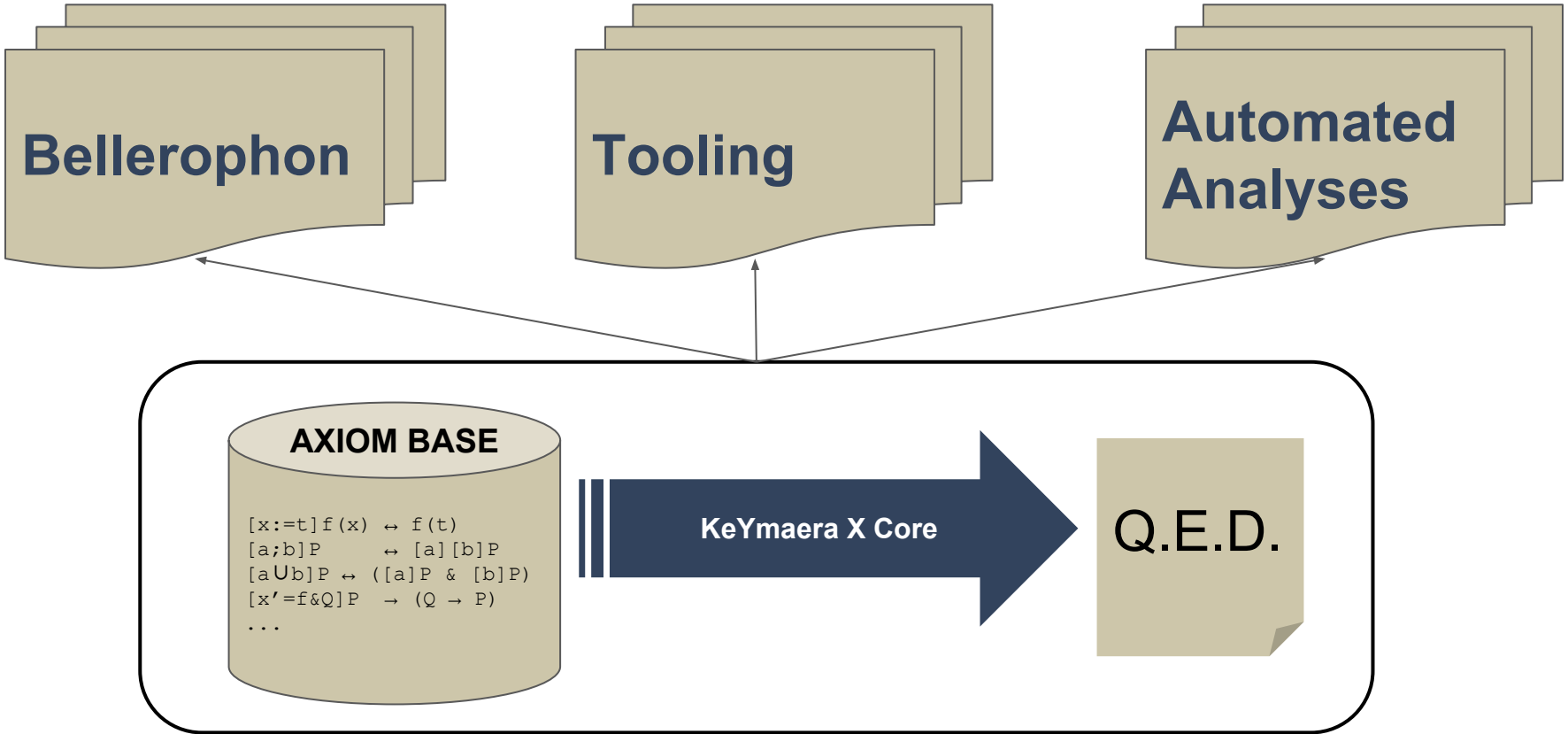
$$[a \cup b]P \leftrightarrow ([a]P \,\&\, [b]P)$$

$$[x'=f\&Q]P \rightarrow (Q \rightarrow P)$$

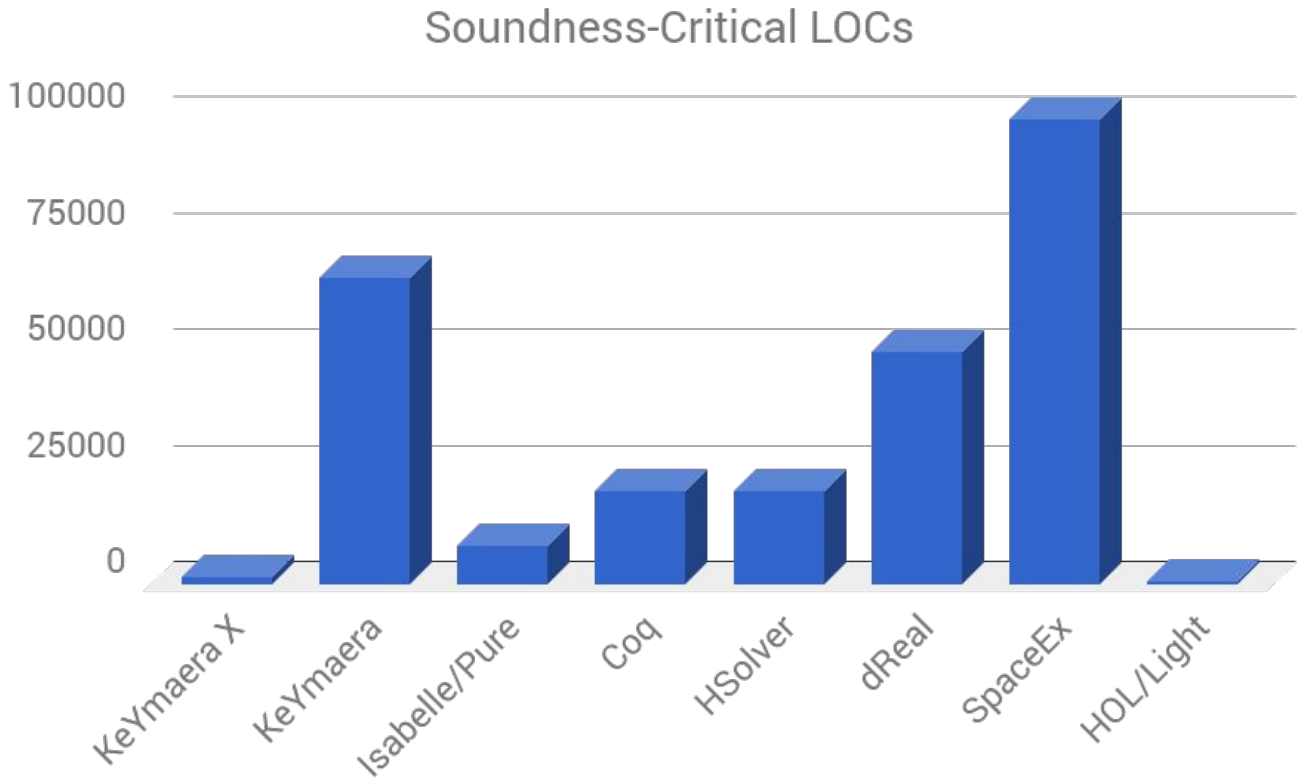
...

Trusted Core

Trustworthy Implementations



Prover Core Comparison



Bellerophon

Bellerophon enables interactive verification and tool development:

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- A **standard library** of common proof techniques.

Bellerophon

Bellerophon enables interactive verification and tool development:

- A **standard library** of common proof techniques.
- A **combinator language/library** for **decomposing** theorems and **composing** proof strategies.

Bellerophon

Standard Library

Tactic	Meaning
`prop`	Applies propositional reasoning exhaustively.
`unfold`	Symbolically executes discrete, loop-free programs.
`loop(J, i)`	Applies loop invariance axiom to position i.
`dI,dG,dC,dW`	Reasoning principles for differential equations.

Bellerophon

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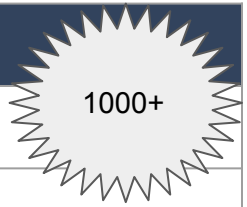
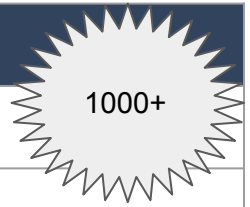


Bellerophon

Combinators

Tactic	Meaning
`prop`	Applies propositional reasoning exhaustively.
`unfold`	Symbolically executes discrete, loop-free programs.
`loop(J, i)`	Applies loop invariance axiom to position i, extends J with constants.
`dI,dG,dC,dW`	Reasoning principles for differential equations.



Combinator	Meaning
A ; B	Execute A on current goal, then execute B on the result.
A | B	Try executing A on current goal. If A fails, execute B on current goal.
A*	Run A until it no longer applies.
A<(B_1,B_2, … ,B_N)	Execute A on current goal to create N subgoals. Run B_i on subgoal i.

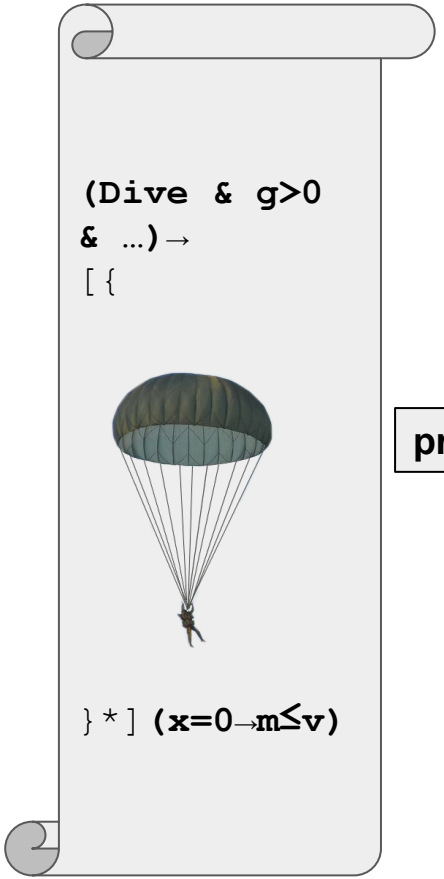
Bellerophon

Isolating Interesting Questions

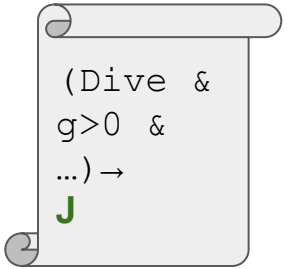
$$(\text{Dive} \;\&\; g>0 \;\&\; \ldots) \rightarrow [\{$$

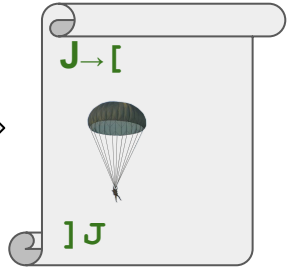
$$\}^*] (x=0 \rightarrow m \leq v)$$

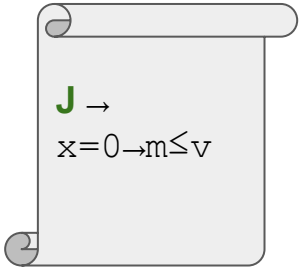
Isolating Interesting Questions



prop ; loop(J,1)







Loop invariant holds initially

Loop invariant is preserved

Loop invariant implies safety

Isolating Interesting Questions

(Dive & g>0 & …)→
[{

}*] (x=0→m≤v)

prop ; loop(J,1)

(Dive & g>0 & …)→ **J**

J→[

]J

J → x=0→m≤v

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Loop invariant is preserved

Loop invariant implies safety

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unfold

J & Dive & r=r_a→ [x′=v,v′=...]J

J & r=r_p→ [x′=v,v′=...]J

Isolating Interesting Questions

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unfold

J & Dive & r=r$_a$→ [x′=v,v′=...]J

J & r=r$_p$→ [x′=v,v′=...]J

Isolating Interesting Questions

```
prop ;  loop(J, 1) <(
   QE, /* Real arith. solver */
   QE,
   unfold ; <(
      … /* parachute open case */
      … /* parachute closed case */
   )
)
```

Trustworthy Standard Library at High Abstraction Level

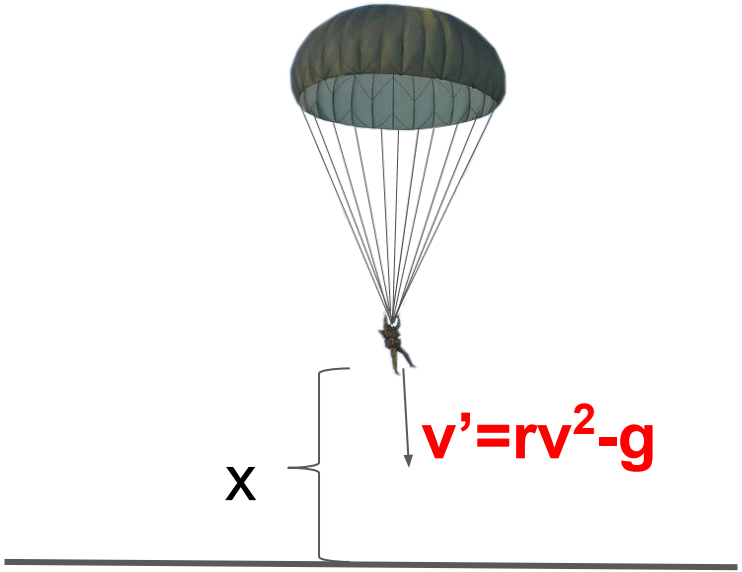
$J \rightarrow [\{ctrl;\ plant\}^*]J$

$J = v > -\sqrt{g/pr} > m\ \&\ \ldots$

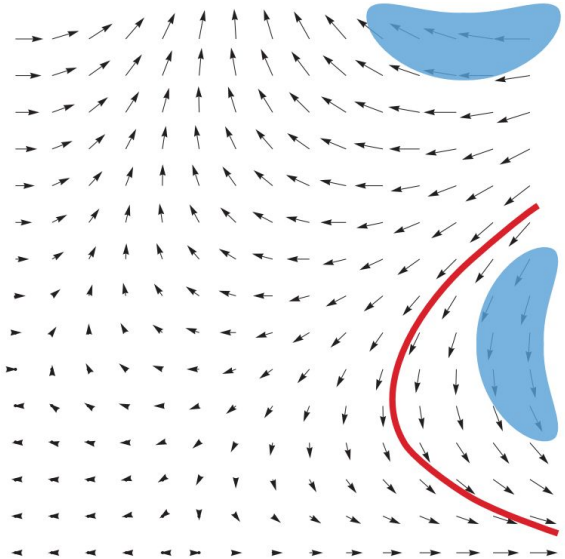
Parachute Open Case:

$$v \geq v_0 - gt \geq v_0 - gT > -\sqrt{g/pr}$$

Inductive invariants



From Axioms to Proof Steps



DI Axiom:

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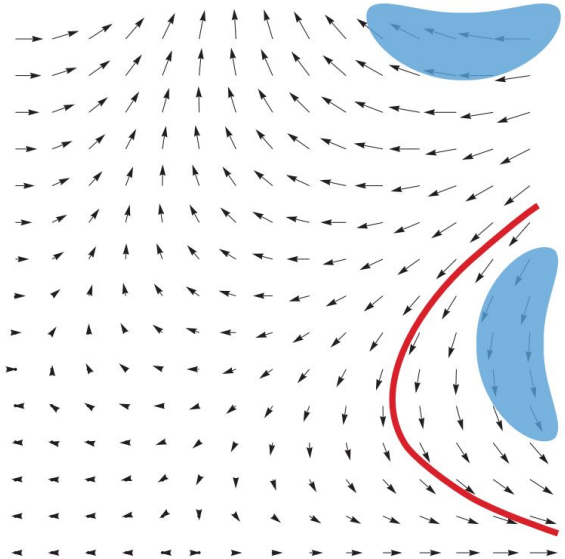
From Axioms to Proof Steps



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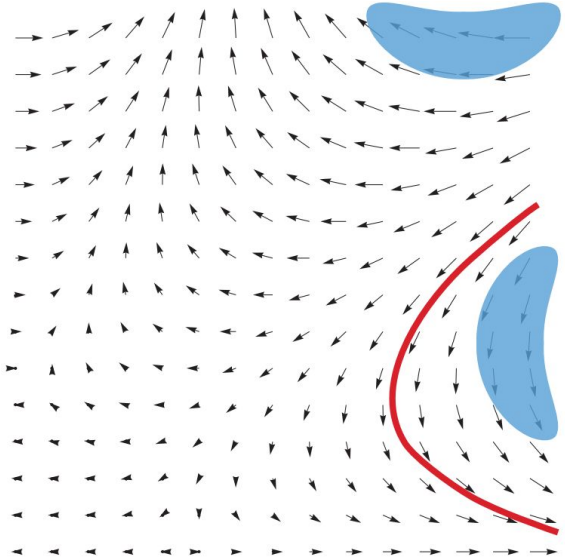

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$$r_p \geq 0$$

From Axioms to Proof Steps

dI Tactic:

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Side derivation:

$$(v \geq v_0 - gt)' \quad \leftrightarrow$$

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$$(v)' \geq (v_0 - gt)' \quad \leftrightarrow$$

$$(v)' \geq (v_0)' - (gt)' \quad \leftrightarrow$$

$$(v)' \geq (v_0)' - (t(g)'+g(t')) \quad \leftrightarrow$$

$$V' \geq v_0' - (tg'+gt')$$

$H=r_p \geq 0 \;\&\; r_a \geq 0 \;\&\; g>0 \;\&\; \ldots$

Automation and Tooling

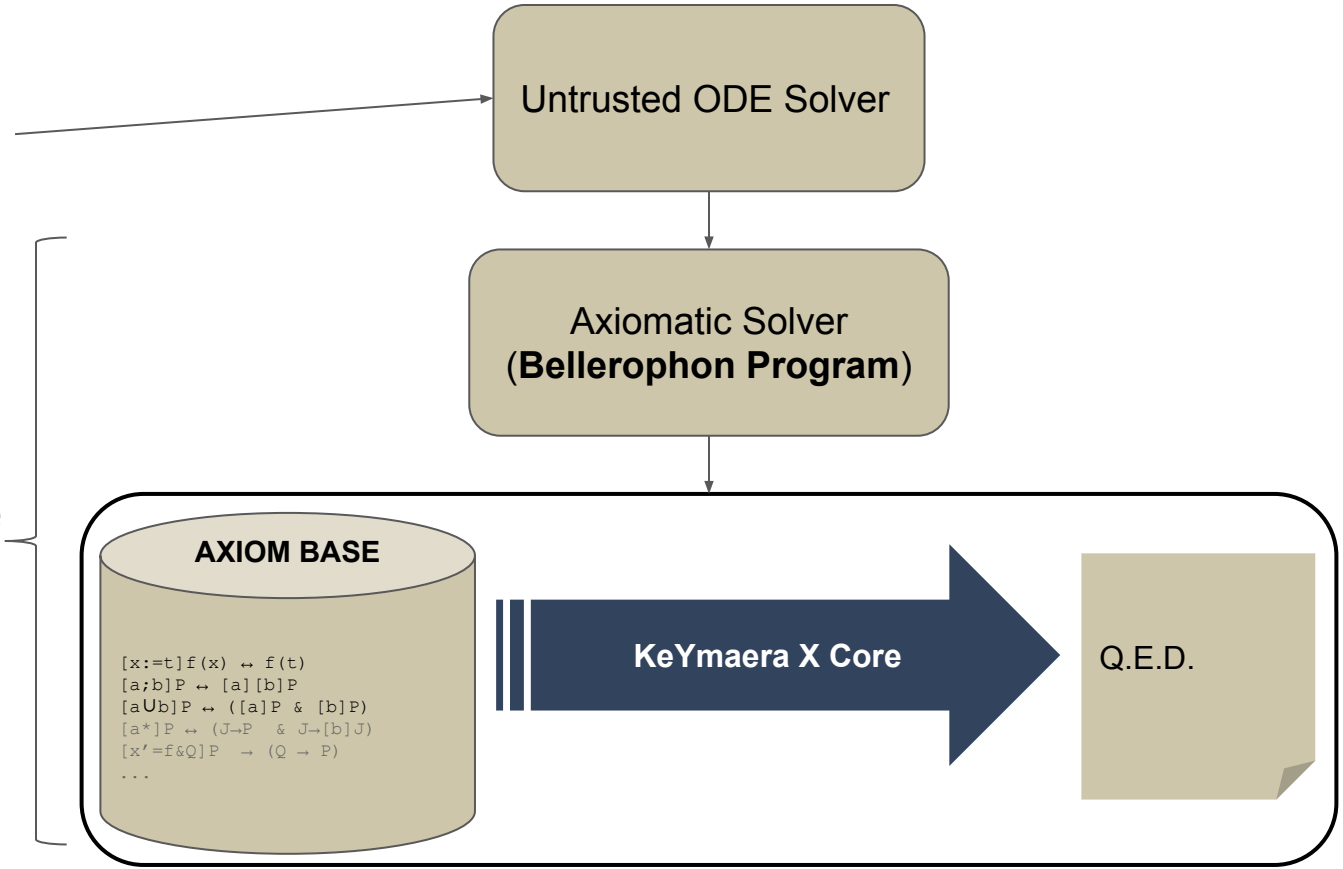
Hybrid Systems Analyses can be built on top of KeYmaera X.

Examples:

- ODE Solver
- Runtime Monitoring

Solving Differential Equations

1. Use untrusted code to find a conjecture.
2. Prove the conjecture systematically, leveraging standard library.



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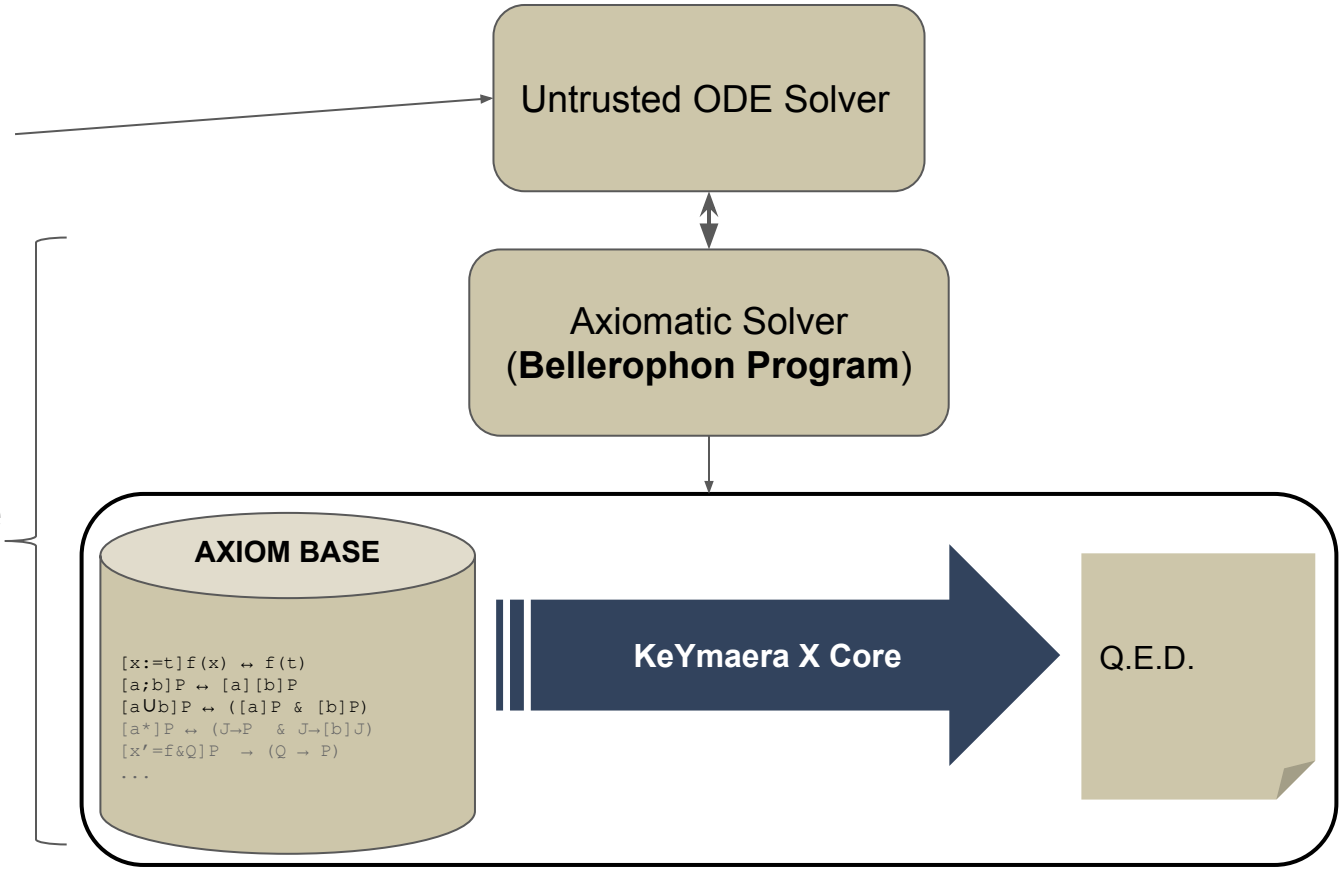
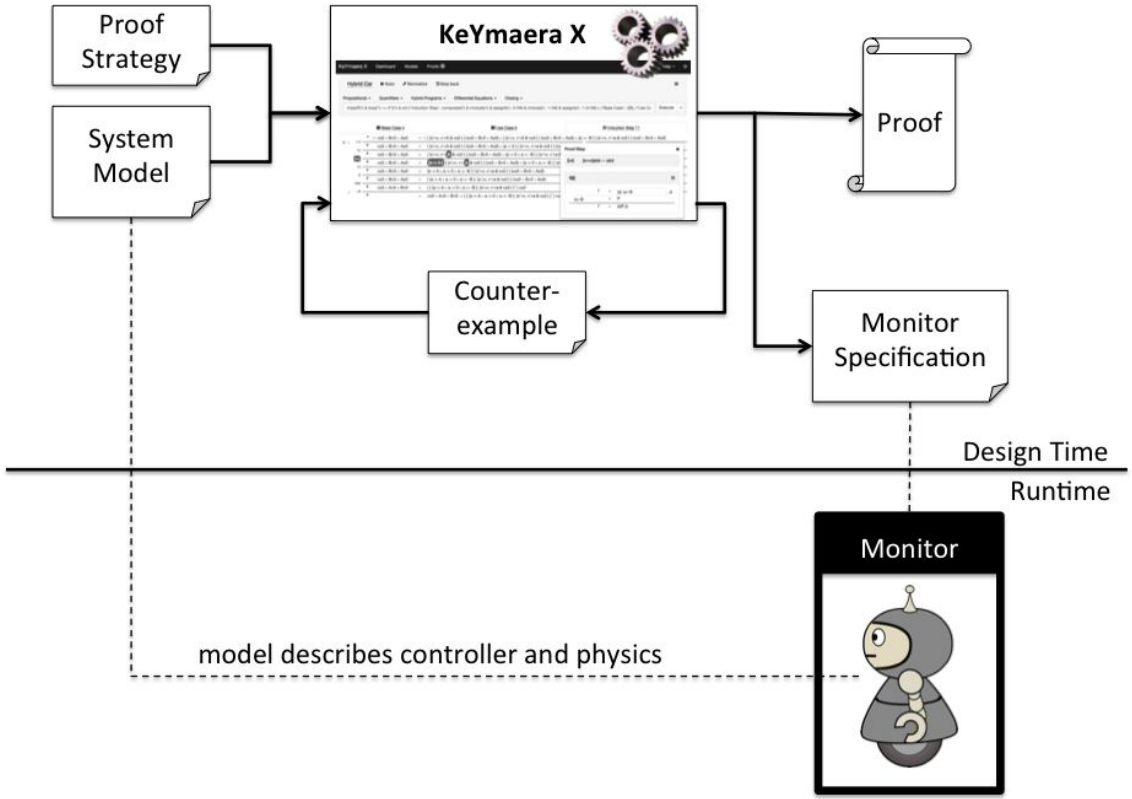
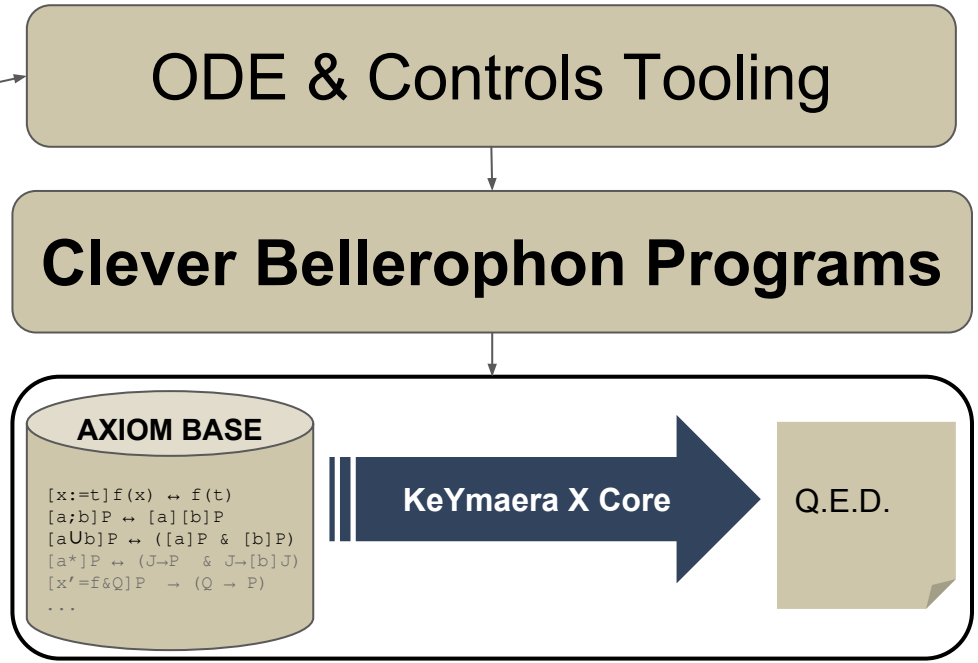


ModelPlex Tactic



Other Proof Automation & Tooling

- **Taylor Series**
- **Bifurcations**
- Limit Cycles
- Numerical tools
- ...



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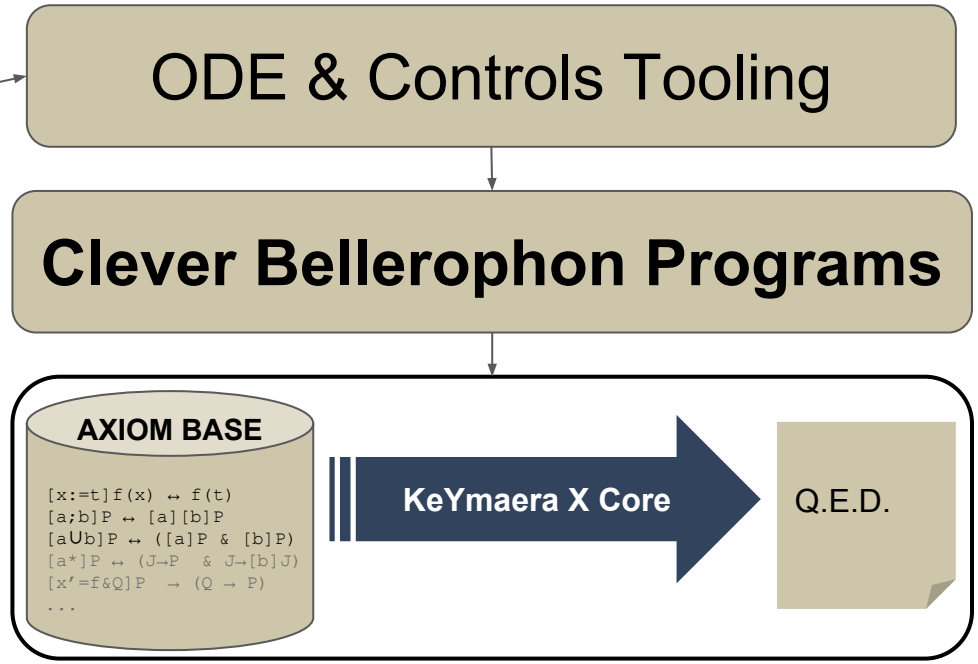
Other Tooling:

- Component-based Verification
- Web UI



Conclusion

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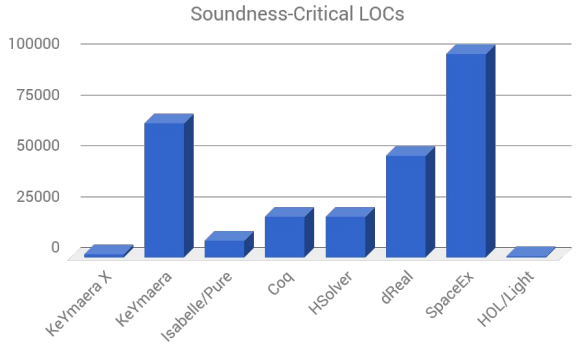

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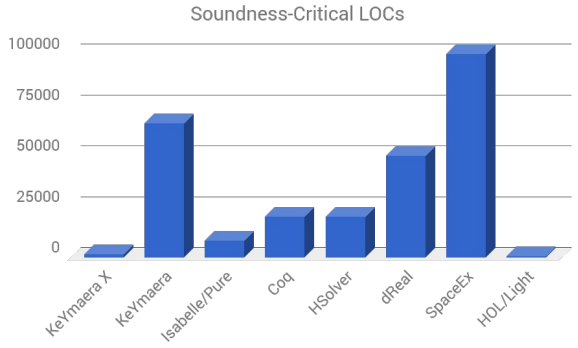

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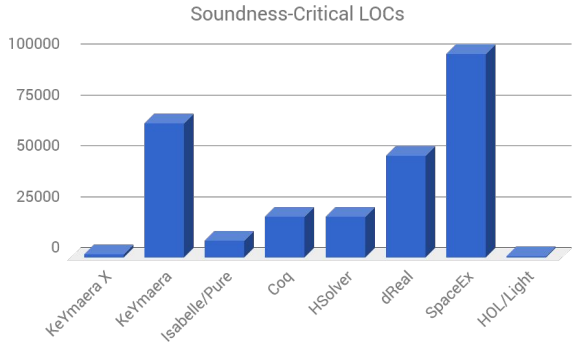

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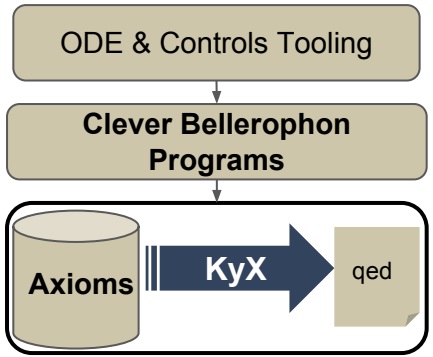

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Project Website (start here)	keymaeraX.org
Online Demo	web.keymaeraX.org
Open Source (GPL)	github.com/ls-lab/KeYmaeraX-release

Thanks: 15-424 students, **Jean-Baptiste Jeannin**, Khalil Ghorbal, **Yanni Kouskoulas** et al., and many others!

Developers:

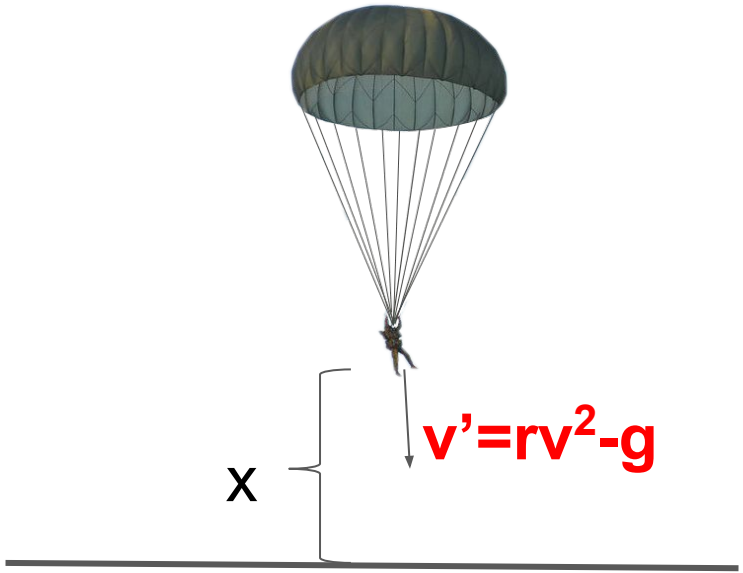
- Stefan Mitsch
- Nathan Fulton
- André Platzer
- Brandon Bohrer
- Jan-David Quesel
- Yong Kiam Tan
- Markus Völp

Differential Ghosts

Parachute Closed:

$$J \;\&\; t=0 \;\&\; r=r_p \rightarrow [x'=v, v'=rv^2-g \;\&\; 0 \leq x \;\&\; t \leq T] v > -\mathrm{sqrt}(g/pr) > m$$

Proof requires a **differential ghost** because the property is **not inductive**.



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An example differential ghost.

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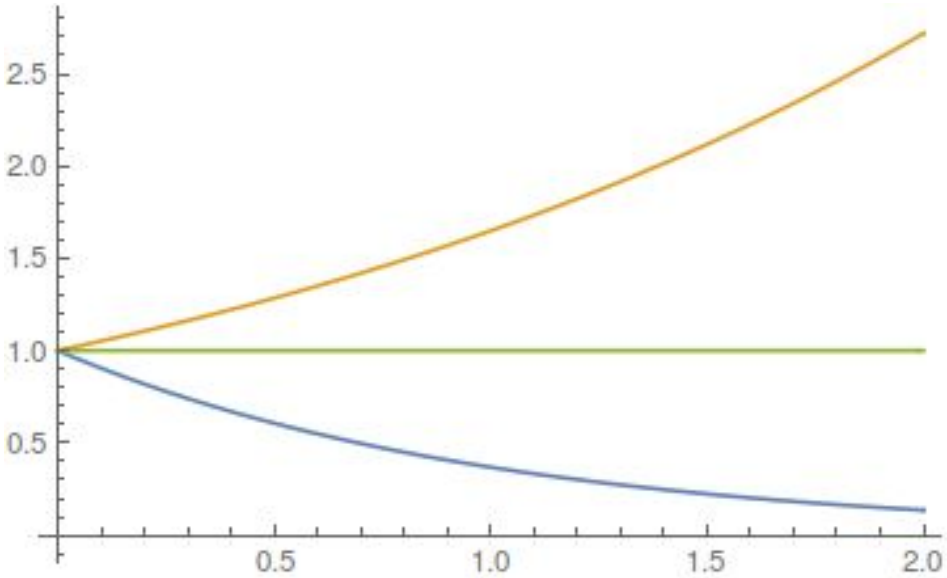

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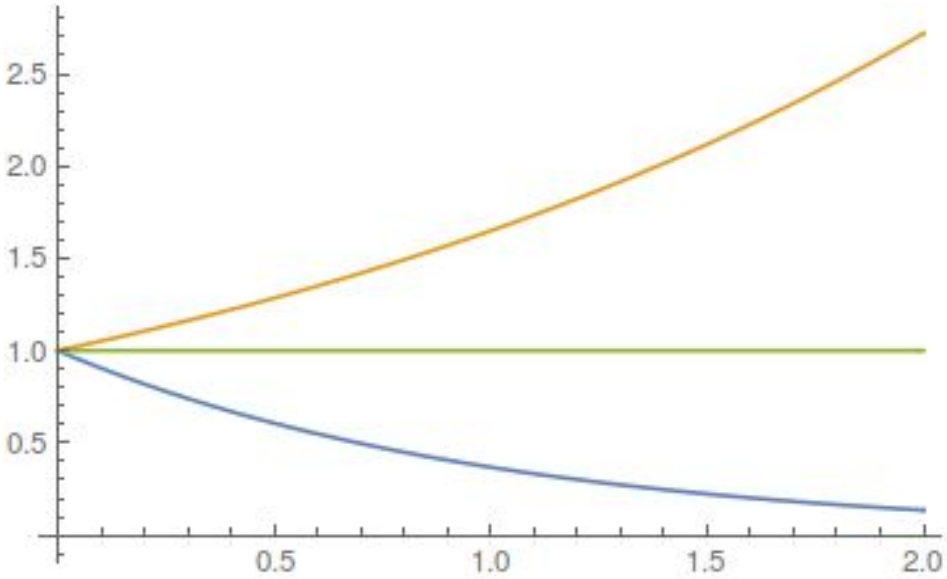

Notice:

$x>0 \leftrightarrow \exists y.\, 1=xy^2$

Therefore, suffices to show:

$1=xy^2 \rightarrow \exists y.\, [x'=-x, y'=y/2]1=xy^2$

Prover Core Comparison

Tool	Trusted LOC (approx.)
KeYmaera X	**1,682 (out of 100,000+)**
KeYmaera	65,989
Isabelle/Pure	8,113
Coq	20,000
HSolver	20,000
dReal	50,000
SpaceEx	100,000